Differential Kinematics

- Relationship between the joint velocities and the corresponding end-effector linear and angular velocity
  - Differential kinematics
  - Geometric Jacobian
  - Jacobian of typical manipulator structures
  - Kinematic singularities
  - Analysis of redundancy
  - Use of redundancy
  - Inverse differential kinematics
  - Analytical Jacobian
  - …
  - …

The Use of Redundancy

$$g'(\dot{\boldsymbol{q}}) = \frac{1}{2}(\dot{\boldsymbol{q}}^T - \dot{\boldsymbol{q}}_a^T)(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_a)$$

$$g'(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \frac{1}{2}(\dot{\boldsymbol{q}}^T - \dot{\boldsymbol{q}}_a^T)(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_a) + \boldsymbol{\lambda}^T(\boldsymbol{v} - \boldsymbol{J}\dot{\boldsymbol{q}})$$

- Optimal solution

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^\dagger \boldsymbol{v} + (\boldsymbol{I} - \boldsymbol{J}^\dagger \boldsymbol{J})\dot{\boldsymbol{q}}_a$$

- Generate *internal motions*

$$\dot{\boldsymbol{q}}_a = k_a \left(\frac{\partial w(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T$$

- *Manipulability measure*

$$w(\boldsymbol{q}) = \sqrt{\det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)}$$

- *Distance from mechanical joint limits*

$$w(\boldsymbol{q}) = -\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}}\right)^2$$

- *Distance from an obstacle*

$$w(\boldsymbol{q}) = \min_{\boldsymbol{p},\boldsymbol{o}} \|\boldsymbol{p}(\boldsymbol{q}) - \boldsymbol{o}\|$$

Kinematic Singularities

- The previous solutions can be computed only when the Jacobian has full rank
- If the manipulator is at a singular configuration, rows of $\boldsymbol{J}$ are linearly dependent
  - If $\boldsymbol{v} \in \mathcal{R}(\boldsymbol{J}) \implies$ a solution $\dot{\boldsymbol{q}}$ by extracting all the linearly independent equations
  - If $\boldsymbol{v} \notin \mathcal{R}(\boldsymbol{J}) \implies$ the system of equations has no solution, the operational space path cannot be executed
- The inversion of the Jacobian at a singularity
  - Small $\det(\boldsymbol{J}) \implies \dot{\boldsymbol{q}}$ high
- *Damped least-squares (DLS) inverse* $\boldsymbol{J}^\star = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T + k^2\boldsymbol{I})^{-1}$

  where $\dot{\boldsymbol{q}}$ minimize

$$g''(\dot{\boldsymbol{q}}) = \|\boldsymbol{v} - \boldsymbol{J}\dot{\boldsymbol{q}}\|^2 + k^2\|\dot{\boldsymbol{q}}\|^2$$

Analytical Jacobian

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q})$$
$$\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{q})$$

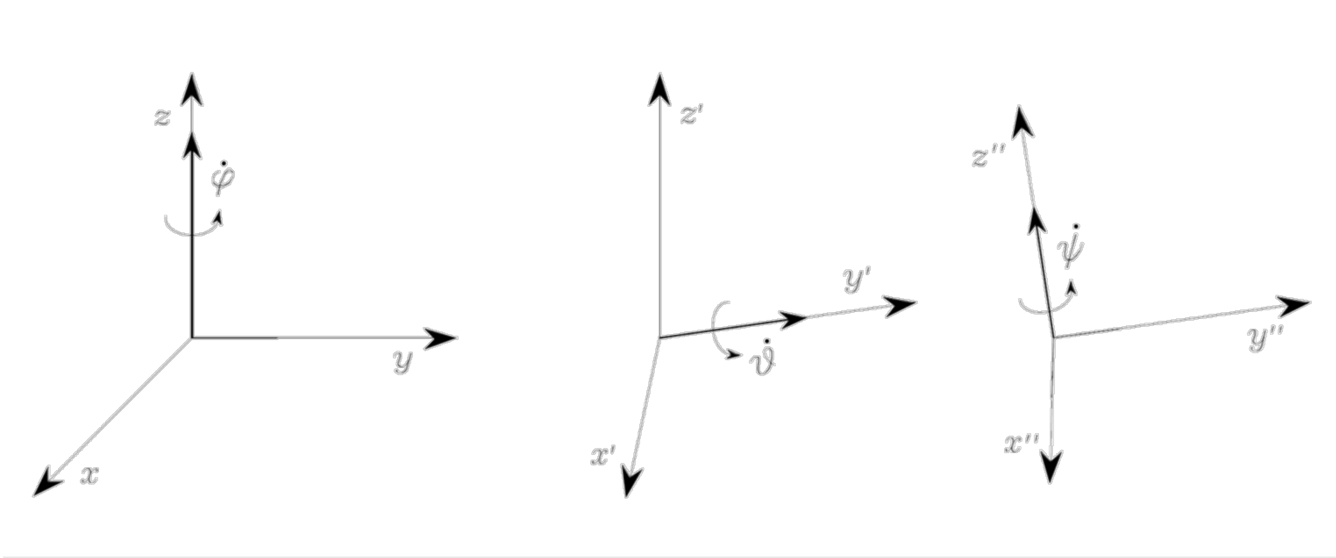
$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{\phi}} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}_\phi(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_P(\boldsymbol{q}) \\ \boldsymbol{J}_\phi(\boldsymbol{q}) \end{bmatrix} \dot{\boldsymbol{q}}$$
$$= \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$$

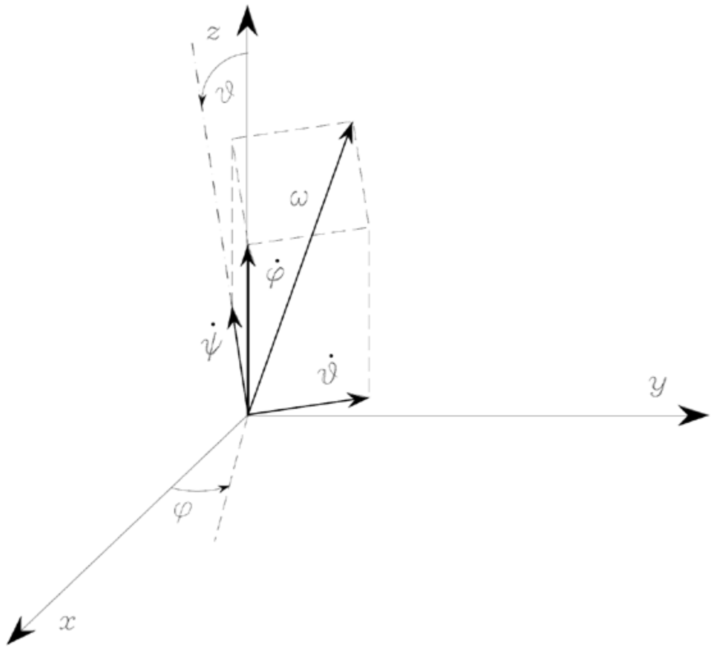
$$\boldsymbol{J}_A(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}(\boldsymbol{q})}{\partial \boldsymbol{q}}$$

Rotational Velocities of Euler Angles ZYZ



- As a result of $\dot{\varphi}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\varphi}\,[0 \quad 0 \quad 1]^T$
- As a result of $\dot{\vartheta}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\vartheta}\,[-s_\varphi \quad c_\varphi \quad 0]^T$
- As a result of $\dot{\psi}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\psi}\,[c_\varphi s_\vartheta \quad s_\varphi s_\vartheta \quad c_\vartheta]^T$

Composition of Elementary Rotations

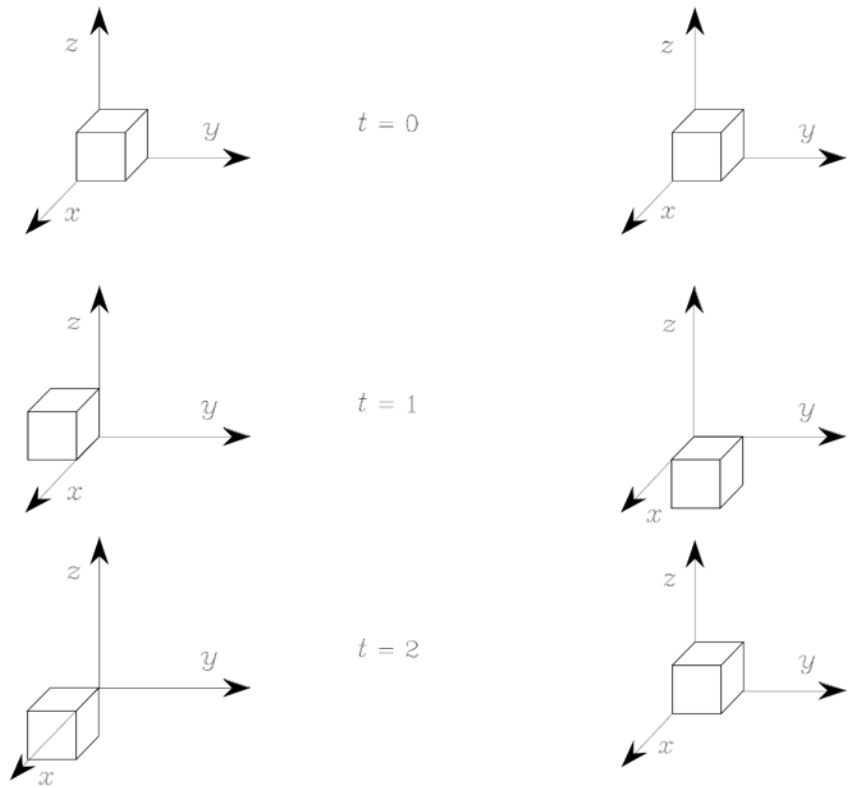


$$\boldsymbol{\omega} = \begin{bmatrix} 0 & -s_{\varphi} & c_{\varphi}s_{\vartheta} \\ 0 & c_{\varphi} & s_{\varphi}s_{\vartheta} \\ 1 & 0 & c_{\vartheta} \end{bmatrix} \dot{\boldsymbol{\phi}} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$$

Nonuniqueness of Orientation

$$\boldsymbol{\omega} = [\pi/2 \quad 0 \quad 0]^T \quad 0 \le t \le 1$$
$$\boldsymbol{\omega} = [0 \quad \pi/2 \quad 0]^T \quad 1 < t \le 2$$

$$\boldsymbol{\omega} = [0 \quad \pi/2 \quad 0]^T \quad 0 \le t \le 1$$
$$\boldsymbol{\omega} = [\pi/2 \quad 0 \quad 0]^T \quad 1 < t \le 2$$



$$\int_0^2 \boldsymbol{\omega} dt = [\pi/2 \quad \pi/2 \quad 0]^T$$

Analytical vs Geometrical Jacobian

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{T}(\boldsymbol{\phi}) \end{bmatrix} \dot{\boldsymbol{x}} = \boldsymbol{T}_A(\boldsymbol{\phi})\dot{\boldsymbol{x}}$$

$$\boldsymbol{J} = \boldsymbol{T}_A(\boldsymbol{\phi})\boldsymbol{J}_A$$

- Geometrical Jacobian
  - referring to quantities of clear physical meaning
- Analytical Jacobian
  - referring to differential quantities of variables defined in the operational space