Differential Kinematics

- Relationship between the joint velocities and the corresponding end-effector linear and angular velocity
  - Differential kinematics
  - Geometric Jacobian
  - Jacobian of typical manipulator structures
  - Kinematic singularities
  - Analysis of redundancy
  - …
  - …

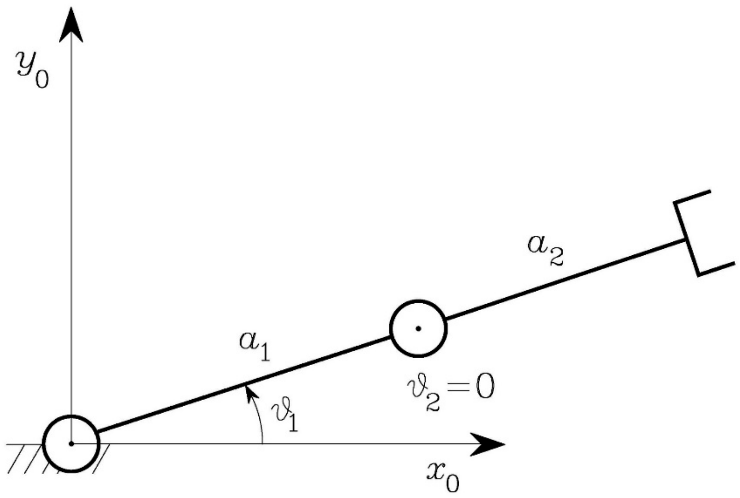
Kinematic Singularities

- Configurations at which $\boldsymbol{J}$ is rank-deficient are termed *kinematic singularities*

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

  - mobility of the structure
  - infinite solutions to the inverse kinematics problem may exist
  - in the neighbourhood of a singularity, small velocities in the operational space may cause large velocities in the joint space
- Classification
  - *Boundary* singularities that occur when the manipulator is either outstretched or retracted
  - *Internal* singularities that occur inside the reachable workspace and are generally caused by the alignment of two or more axes of motion

Two-link Planar Arm



$$\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

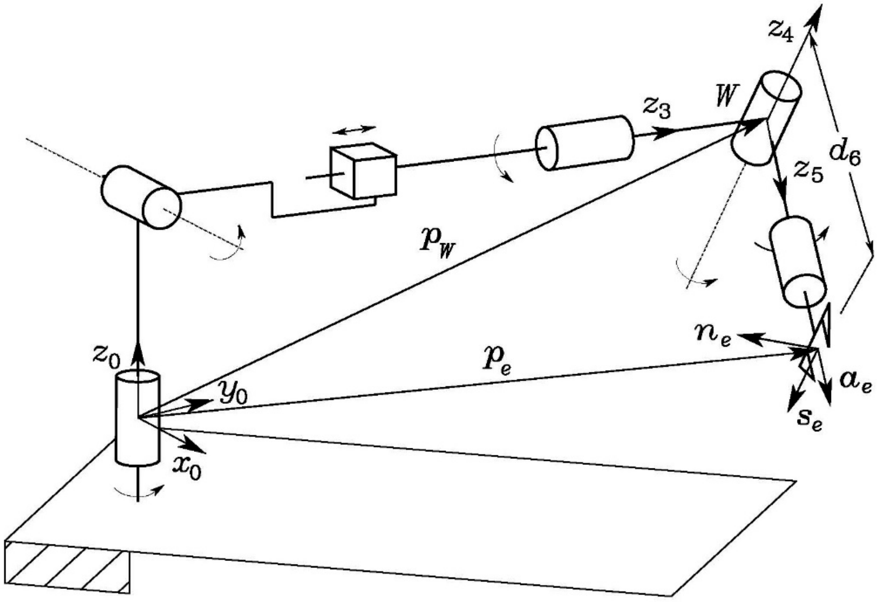
$$\det(\boldsymbol{J}) = a_1 a_2 s_2$$

$$\Downarrow$$

$$\vartheta_2 = 0 \qquad \vartheta_2 = \pi$$

- Two column vectors $[-(a_1 + a_2)s_1 \; (a_1 + a_2)c_1]^T$ and $[-a_2 s_1 \; a_2 c_1]^T$ of the Jacobian become parallel
- The tip velocity components are not independent

Singularity Decoupling



- Computation of *arm singularities* resulting from the motion of the first 3 or more links
- Computation of *wrist singularities* resulting from the motion of the wrist joints

Singularity Decoupling

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} \end{bmatrix}$$

$$\boldsymbol{J}_{12} = \begin{bmatrix} \boldsymbol{z}_3 \times (\boldsymbol{p} - \boldsymbol{p}_3) & \boldsymbol{z}_4 \times (\boldsymbol{p} - \boldsymbol{p}_4) & \boldsymbol{z}_5 \times (\boldsymbol{p} - \boldsymbol{p}_5) \end{bmatrix}$$

$$\boldsymbol{J}_{22} = \begin{bmatrix} \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}$$

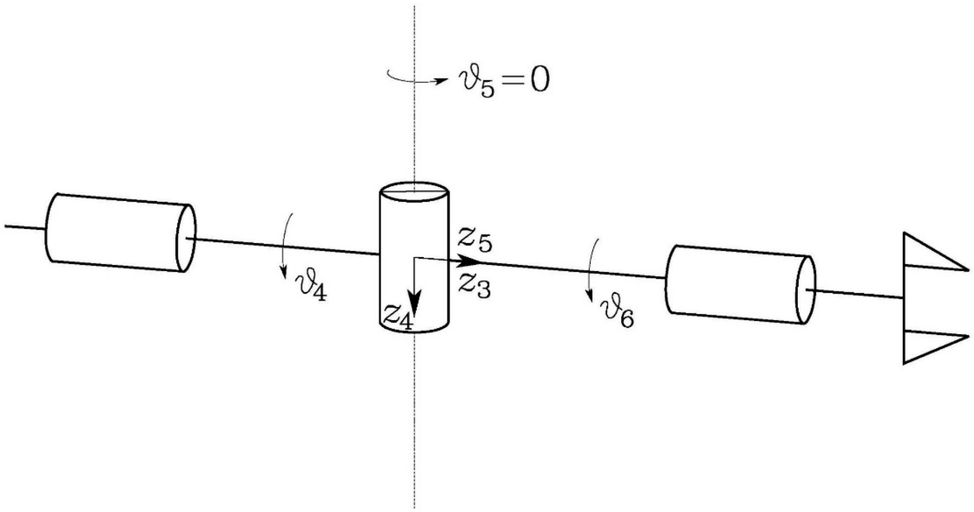
- $\boldsymbol{p} = \boldsymbol{p}_W \implies \boldsymbol{p}_W - \boldsymbol{p}_i$ parallel to $\boldsymbol{z}_i, i = 3, 4, 5$

$$\boldsymbol{J}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\det(\boldsymbol{J}) = \det(\boldsymbol{J}_{11})\det(\boldsymbol{J}_{22})$$

$$\det(\boldsymbol{J}_{11}) = 0 \qquad \det(\boldsymbol{J}_{22}) = 0$$

Wrist Singularities



$$\vartheta_5 = 0 \qquad \vartheta_5 = \pi$$

- Rotations of equal magnitude about opposite directions on ϑ_4 and ϑ_6 do not produce any end-effector rotation

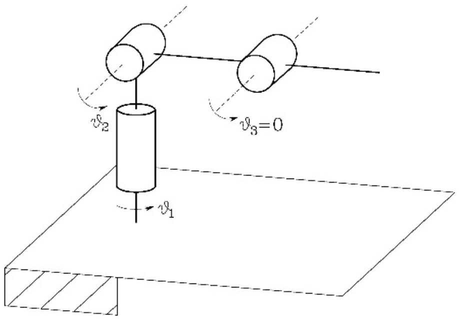
Arm Singularities

- Anthropomorphic arm

$$\det(\boldsymbol{J}_P) = -a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

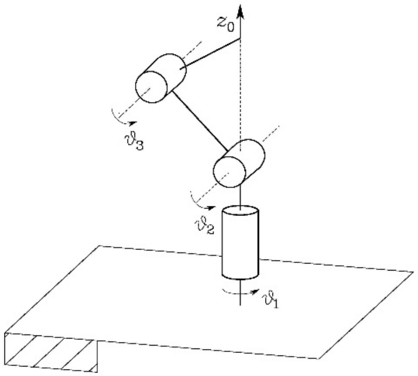
$$s_3 = 0 \qquad a_2 c_2 + a_3 c_{23} = 0$$

- Elbow singularity



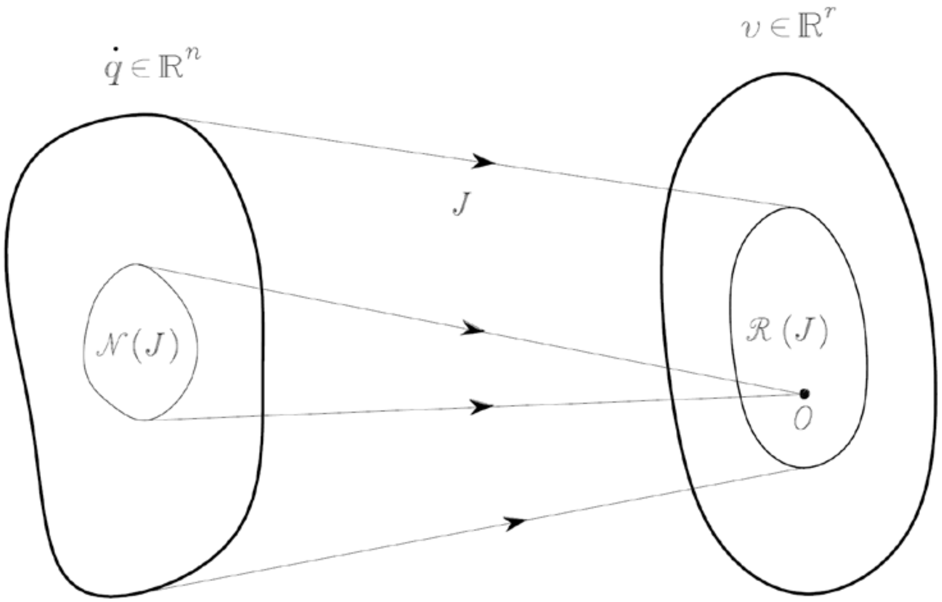
$$\vartheta_3 = 0 \qquad \vartheta_3 = \pi$$

- Shoulder singularity



$$p_x = p_y = 0$$

Analysis of Redundancy



$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- if $\varrho(\boldsymbol{J}) = r$

$$\dim\big(\mathcal{R}(\boldsymbol{J})\big) = r \qquad \dim\big(\mathcal{N}(\boldsymbol{J})\big) = n - r$$

- in general $\dim\big(\mathcal{R}(\boldsymbol{J})\big) + \dim\big(\mathcal{N}(\boldsymbol{J})\big) = n$

Analysis of Redundancy

- If $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^* + \boldsymbol{P}\dot{\boldsymbol{q}}_a$$

  where $\mathcal{R}(\boldsymbol{P}) \equiv \mathcal{N}(\boldsymbol{J})$

- Indeed $\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{J}\dot{\boldsymbol{q}}^* + \boldsymbol{J}\boldsymbol{P}\dot{\boldsymbol{q}}_a = \boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v}$

  $\dot{\boldsymbol{q}}_a$ generates internal motions

Inverse Differential Kinematics

- Direct kinematics = Non-linear equation
- Differential kinematics equation = Linear mapping between the joint velocity space and the operational velocity space
- Assigned $\boldsymbol{v}(t)$ and the initial conditions on position and orientation

$$\Longrightarrow \quad (\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$$

- if n=r

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v}$$

$$\boldsymbol{q}(t) = \int_0^t \dot{\boldsymbol{q}}(\varsigma)d\varsigma + \boldsymbol{q}(0)$$

- Euler integration method

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \dot{\boldsymbol{q}}(t_k)\Delta t$$

Redundant Manipulators

- For a given configuration $\boldsymbol{q}$, find solutions $\dot{\boldsymbol{q}}$ that satisfy $\boldsymbol{v} = \boldsymbol{J}\dot{\boldsymbol{q}}$ and *minimize* the quadratic cost functional of joint velocities

$$g(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}}$$

- *Method of Lagrange multipliers*

$$g(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}} + \boldsymbol{\lambda}^T (\boldsymbol{v} - \boldsymbol{J}\dot{\boldsymbol{q}})$$

$$\left(\frac{\partial g}{\partial \dot{\boldsymbol{q}}}\right)^T = \mathbf{0} \qquad \left(\frac{\partial g}{\partial \boldsymbol{\lambda}}\right)^T = \mathbf{0}$$

Redundant Manipulators

- Optimal solution $\dot{\boldsymbol{q}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{v}$

if $\boldsymbol{W} = \boldsymbol{I}$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^\dagger \boldsymbol{v}$$

where $\boldsymbol{J}^\dagger = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$

is the *right pseudo-inverse* of $\boldsymbol{J}$