Scuola di Dottorato CIRA
Controllo di Sistemi Robotici per la Manipolazione e la Cooperazione
Bertinoro (FC), 14–16 Luglio 2003

Robots with Flexible Links: Modeling and Control

Alessandro De Luca
Dipartimento di Informatica e Sistemistica (DIS)
Università di Roma "La Sapienza"

Outline

- Motivation for considering distributed link flexibility
- Dynamic modeling of FL robots: single link and multiple link cases
- Formulation of control problems
- Controllers for regulation tasks
- Controllers for joint and end-effector trajectory tracking tasks
- Controllers for rest-to-rest motion tasks
- Conclusions
- References

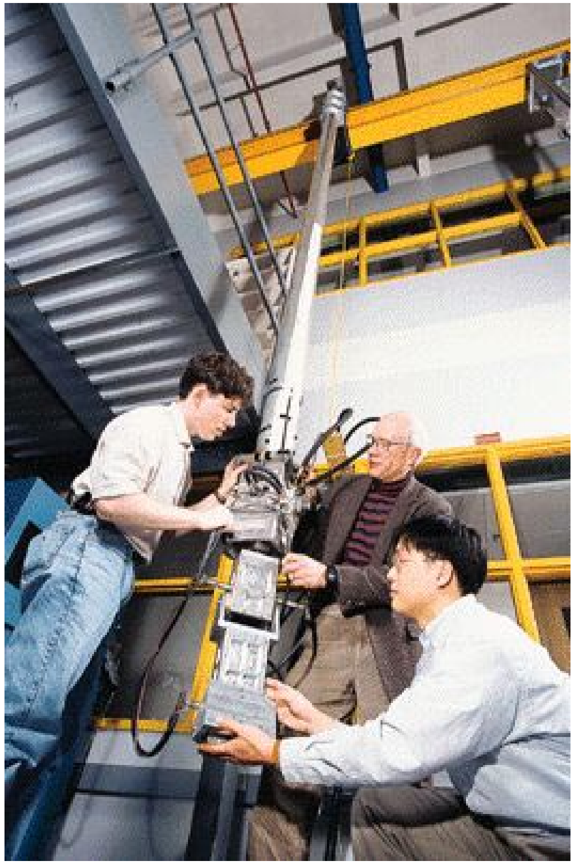
Motivation

- distributed link deformation in robot manipulators arises when
  - the design of very long and slender arms is needed for the application
  - lightweight materials are used (without special care)
- ‘link rigidity’ is always an ideal assumption and may fail when increasing
  - payload-to-weight ratio
  - motion speed
  - control bandwidth
- flexible structures in motion are present in different domains: space manipulators, robots for underwater and underground waste sites, automated cranes, . . .
- as for joint elasticity, neglected link flexibility limits static (steady-state error) or dynamic (vibrations, poor tracking) task performance
- from the control point of view, there is an additional problem of non-colocation between input commands and typical outputs to be controlled

Robots with link flexibility — SSRMS

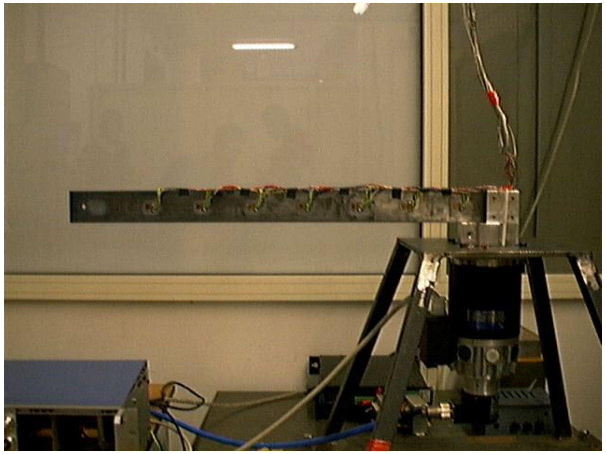
- Space Shuttle Remote Manipulation System (Canadarm) → telemanipulated by astronauts
- link bending due to fast motion (not gravity!)

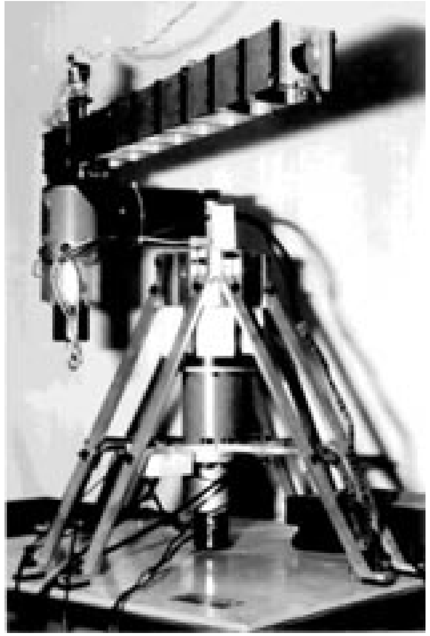
Robots with link flexibility — Sam II



- developed at Georgia Tech (W. Book)
- macro-micro concept for remote exploration and manipulation of nuclear waste sites

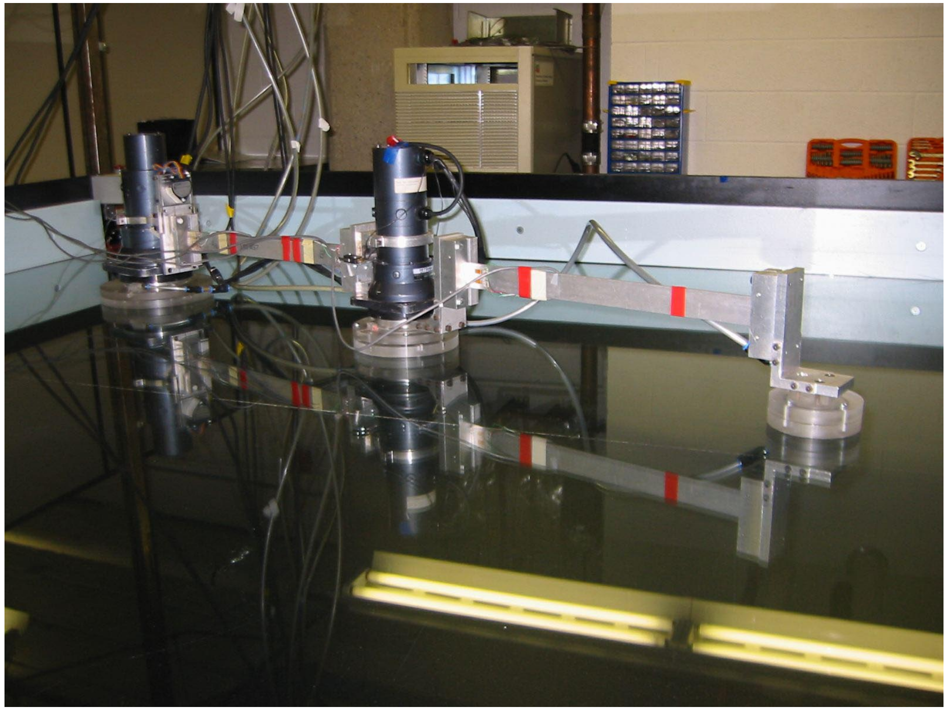
Robots with link flexibility — prototypes in Roma





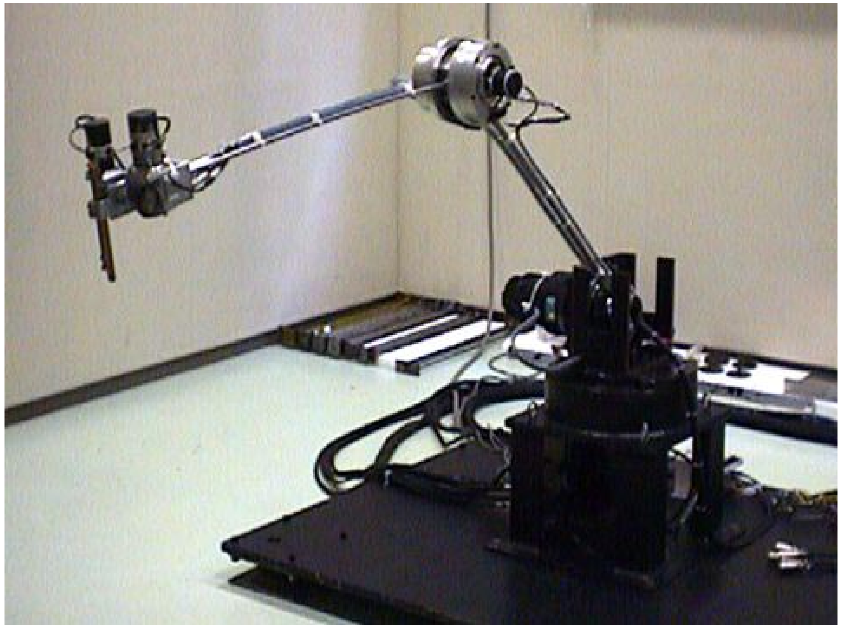
- DMA harmonic steel beam (0.5 kg): DD-DC motor, encoder, 7 strain gauges
- DIS/DIA FLEXARM: planar two-link with flexible forearm (1.8 kg), DD-DC motors, encoders, on-board optical sensor measuring deformation at three points

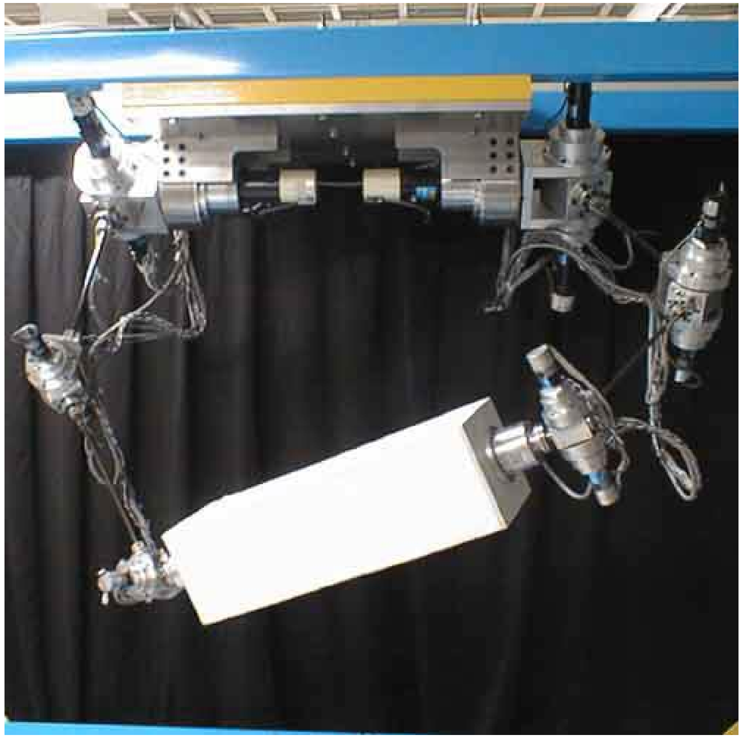
Robots with link flexibility — prototype in Waterloo



- WATFLEX planar 2R with both link flexible (each with 2 strain gauges), moving with air bearings on a glass table to support the weight of the second motor; encoders, tachometers, overviewing CCD camera

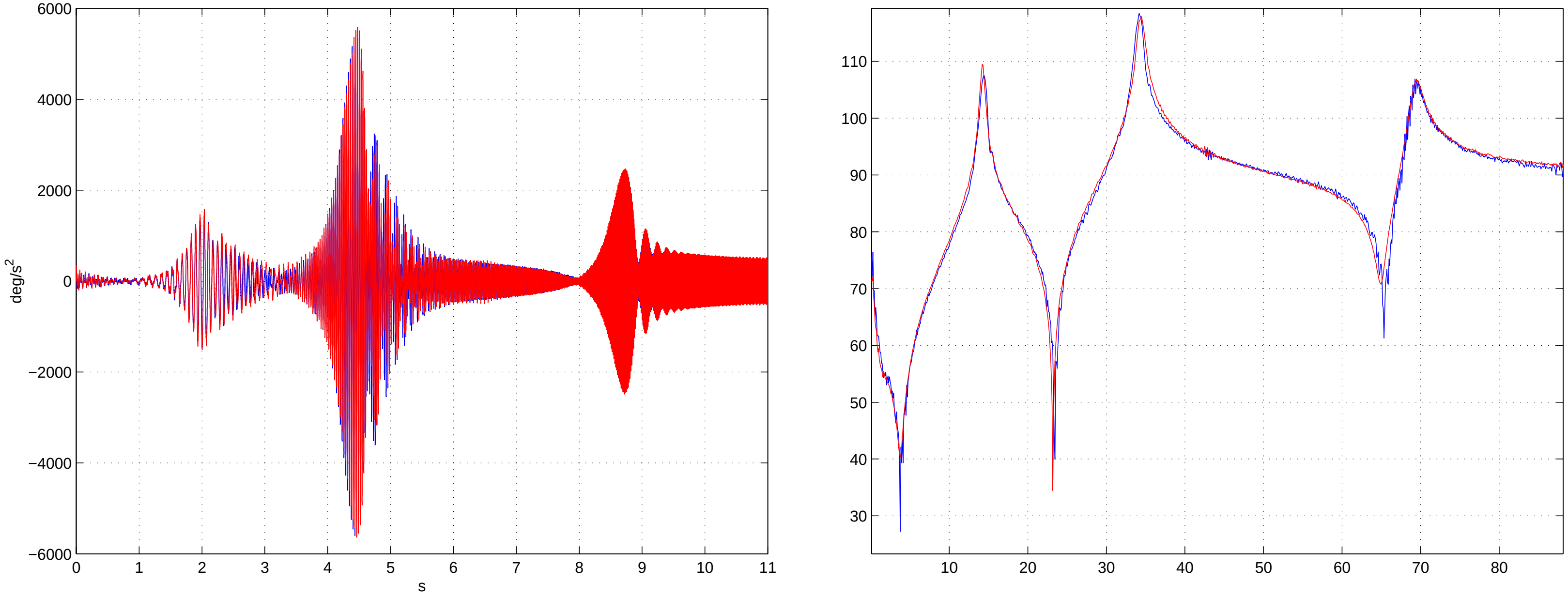
Robots with link flexibility — prototypes in Japan





- spatial 3R flexible arm at Kyoto
- cooperating 6R flexible arms at Tohoku

Frequency identification of a single flexible link

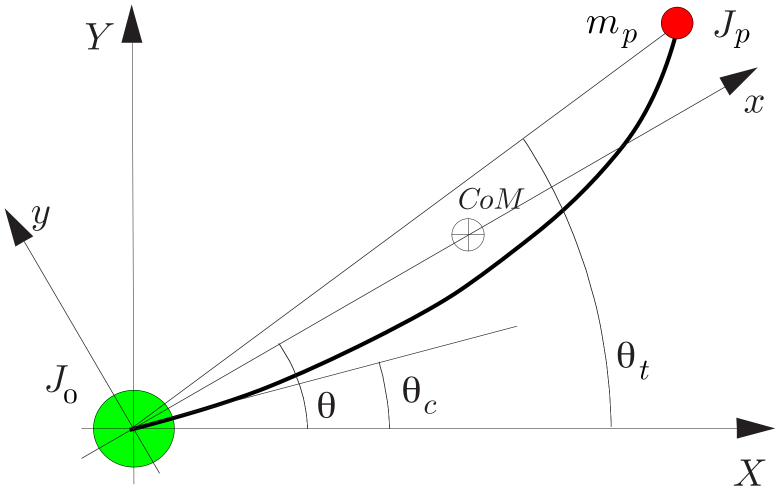


[left] frequency sweep joint acceleration signal: plant vs. model

[right] joint acceleration frequency response: plant vs. model (matching within 1% of resonances at $f_1 = 14.4$, $f_2 = 34.2$, and $f_3 = 69.3$ Hz)

Dynamic modeling of a single flexible link

- one-link flexible arm modeled as a Euler-Bernoulli beam in rotation



- length ℓ, uniform density ρ, Young modulus $\cdot$ cross-section inertia EI
- actuator inertia J_0, payload mass m_p and inertia J_p
- reference frames: (X, Y) absolute; (x, y) moving with instantaneous CoM

Assumptions and definitions

- Euler-Bernoulli theory applies to slender arm design (length vs. section)
- beam undergoes small deformations of pure bending type in the plane of motion (no torsion or compression)
- bending deformation $w(x,t)$, with $x \in [0,\ell]$, is directed along the y direction (no shear)
- rotational inertia of beam sections is neglected ($\rightarrow$ Timoshenko theory) as well as the isoperimetric constraint (‘extension’ of beam neutral axis is negligible)
- definition of other relevant angular variables:
  - position $\theta(t)$ of the CoM (not measurable, but convenient)
  - position $\theta_c(t)$ of the tangent to the link base (measured by motor encoder)
  - position $\theta_t(t)$ of a line pointing to the beam tip (measurable in several ways)

- build the Lagrangian from kinetic and elastic potential energy of the beam
- using Hamilton principle and calculus of variations, the bending deformation $w(x,t)$ and angle $\theta(t)$ satisfy the linear differential equations

$$\boxed{EIw''''(x,t) + \rho(\ddot{w}(x,t) + x\ddot{\theta}(t)) = 0} \qquad \boxed{\tau(t) - J\ddot{\theta}(t) = 0}$$

  i.e., a PDE and an ODE (*rigid motion*), where $J = J_0 + (\rho\ell^3)/3 + J_p + m_p\ell^2$ and $\tau =$ torque input

- geometric/dynamic boundary conditions associated to the PDE

$$\begin{aligned}
w(0,t) &= 0 \\
EIw''(0,t) &= J_0(\ddot{\theta}(t) + \ddot{w}'(0,t)) - \tau(t) && \text{(balance of moments at base)} \\
EIw''(\ell,t) &= -J_p(\ddot{\theta}(t) + \ddot{w}'(\ell,t)) && \text{(balance of moments at tip)} \\
EIw'''(\ell,t) &= m_p(\ell\ddot{\theta}(t) + \ddot{w}(\ell,t)) && \text{(balance of shear forces at tip)}
\end{aligned}$$

- in free evolution ($\tau(t) \equiv 0 \Rightarrow \ddot{\theta}(t) = 0$), PDE solved by separation of variables

$$w(x,t) = \phi(x)\delta(t) \quad \Rightarrow \quad \frac{EI}{\rho}\frac{\phi''''(x)}{\phi(x)} = -\frac{\ddot{\delta}(t)}{\delta(t)} \triangleq \omega^2$$

for a positive constant ω^2 (self-adjoint problem) to be determined

- time solution

$$\ddot{\delta}(t) = -\omega^2\delta(t) \quad \Rightarrow \quad \delta(t) = c_1 \sin \omega t + c_2 \cos \omega t$$

with c_1, c_2 depending on the initial conditions $\delta(0)$ and $\dot{\delta}(0)$

- space solution (try it!)

$$\phi''''(x) = \beta^4\phi(x) \qquad \beta^4 = \frac{\rho\omega^2}{EI}$$

$$\Rightarrow \quad \phi(x) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x$$

with A, B, C, D obtained from the geometric/dynamic b.c.'s on $w(x,t)$

- using $w(x,t)=\phi(x)\delta(t)$ and $\ddot{\delta}=-\omega^2\delta$, and holding the b.c.'s for any $\delta(t)$, these are rewritten in terms of $\phi(x)$ only

$$\begin{aligned} \phi(0) &= 0 \\ EI\phi''(0)+J_0\,\omega^2\phi'(0) &= 0 \\ EI\phi''(\ell)-J_p\,\omega^2\phi'(\ell) &= 0 \\ EI\phi'''(\ell)+m_p\,\omega^2\phi(\ell) &= 0 \end{aligned}$$

- using the general solution $\phi(x)$, a system of linear homogeneus equations follows

$$\left[\quad \mathcal{A}(EI,\rho,\ell,J_0,m_p,J_p,\beta) \quad\right]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=0 \qquad (\star)$$

to exclude the trivial solution, the determinant of matrix $\mathcal{A}$ should be zero (eigenvalue problem)

- $\det \mathcal{A}(\beta) = 0$ at infinite (but countable!) positive increasing roots β_i of the trascendental characteristic equation

$$\boxed{\begin{aligned}&(c\,sh - s\,ch) - \frac{2m_p}{\rho}\beta_i\, s\,sh - \frac{m_p}{\rho^2}\beta_i^4 (J_0 + J_p)(c\,sh - s\,ch) - \frac{2J_p}{\rho}\beta_i^3\, c\,ch \\ &- \frac{J_0}{\rho}\beta_i^3 (1 + c\,ch) + \frac{J_0 J_p}{\rho^2}\beta_i^6 (c\,sh + s\,ch) - \frac{J_0 J_p m_p}{\rho^3}\beta_i^7 (1 - c\,ch) = 0\end{aligned}}$$

where $s = \sin \beta_i \ell$, $c = \cos \beta_i \ell$, $sh = \sinh \beta_i \ell$, $ch = \cosh \beta_i \ell$

- this is an *exact* result, that includes common physical approximations
  - clamped-free model: $m_p = J_p = 0$, $J_0 \to \infty \;\Rightarrow\; \boxed{1 + c\,ch = 0}$
  - pinned-free model: $m_p = J_p = J_0 = 0 \;\Rightarrow\; \boxed{c\,sh - s\,ch = 0}$

- associated to each root β_i we have:
  - an eigenfrequency $\omega_i = \sqrt{EI\beta_i^4/\rho}$, characterizing a system vibration
  - an eigenvector $\phi_i(x)$ —a spatial deformation mode, defined up to a constant
  - a deformation variable $\delta_i(t)$, oscillatory in time
- a finite-dimensional approximation of the distributed bending deformation is obtained by truncation

$$w(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\delta_i(t) \approx \sum_{i=1}^{\boxed{n_e}} \phi_i(x)\delta_i(t)$$

  where n_e is the (arbitrary) number of orthogonal modes that we wish to include
- normalization of the modes can be chosen in different ways (some integral of the $\phi_i(x)$ equal to 1, to m, ...)

- resulting dynamic model is particularly simple (after fairly complex analysis . . .)

$$\boxed{\begin{aligned} J\ddot{\theta} &= \tau \\ \ddot{\delta}_i + \omega_i^2 \delta_i &= \phi_i'(0)\tau \qquad i = 1, \dots, n_e \end{aligned}}$$

- remarkable properties:
  - rigid motion $\theta(t)$ and each vibratory motion $\delta_i(t)$ are decoupled in free evolution ($\tau(t) \equiv 0$)
  - all modes are excited by an input command $\tau(t) \neq 0$, with a weighting that depends on $\phi_i'(0)$ —the tangent at the link base of each deformation mode
  - arm 'stiffness' is summarized by the (squared) eigenfrequencies ω_i
  - each vibratory motion is persistent during free evolution, if initially excited by $\delta_i(0) \neq 0$ (absence of damping)

- modal damping can be easily included in the dynamic model

$$J\ddot{\theta} = \tau$$
$$\ddot{\delta}_i + 2\zeta_i\omega_i\dot{\delta}_i + \omega_i^2\delta_i = \phi_i'(0)\tau \qquad i = 1, \dots, n_e$$

with coefficients $\zeta_i \in [0, 1)$

- its matrix version, with generalized coordinates $q = (\,\theta\ \delta_1\ \dots\ \delta_{n_e}\,)^T \in I\!R^{n_e+1}$, shows the classical *mass-spring-damper* form

$$M\ddot{q} + D\dot{q} + Kq = B\,\tau$$

with

$$M = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2Z\Omega \end{bmatrix} \quad K = \begin{bmatrix} 0 & 0 \\ 0 & \Omega^2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ \Phi' \end{bmatrix}$$

$\Omega = \mathrm{diag}\,\{\omega_1, \dots, \omega_{n_e}\}$, $Z = \mathrm{diag}\,\{\zeta_1, \dots, \zeta_{n_e}\}$, $\Phi' = (\,\phi_1'(0)\ \dots\ \phi_{n_e}'(0)\,)^T$

- a different (but equivalent) choice of generalized coordinates may let the input τ appear in just one equation

$$\begin{aligned} (\theta, \delta) &= (\theta, \delta_1, \ldots, \delta_{n_e}) \\ &\Downarrow \\ (\theta_c, \delta) &= (\theta + \delta^T \Phi', \delta) = (\theta + \sum \phi_i'(0)\delta_i\,, \delta_1, \ldots, \delta_{ne}) \end{aligned}$$

leads to

$$\begin{bmatrix} J & -J\Phi'^T \\ -J\Phi' & I + J^2\Phi'\Phi'^T \end{bmatrix} \begin{bmatrix} \ddot{\theta}_c \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2Z\Omega \end{bmatrix} \begin{bmatrix} \dot{\theta}_c \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Omega^2 \end{bmatrix} \begin{bmatrix} \theta_c \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tau$$

with same (diagonal) damping D and stiffness K matrices, but full inertia matrix

Choice of the controlled output

- joint level angular output (clamped angle)

$$\theta_c = \theta + \sum_{i=1}^{n_e} \phi_i'(0)\,\delta_i$$

!! is always minimum phase (zeros in left-hand side of complex plane)

- tip level angular output

$$\theta_t = \theta + \sum_{i=1}^{n_e} \frac{\phi_i(\ell)}{\ell}\,\delta_i$$

!! is typically non-minimum phase (at least for no tip payload)

- output at a point $x \in [0, \ell]$ along the link

$$\theta_x = \theta + \sum_{i=1}^{n_e} \frac{\phi_i(x)}{x}\,\delta_i$$

Transfer functions

- torque $\tau \mapsto$ clamped joint angle θ_c

$$P_c(s) = \frac{1}{Js^2} + \sum_{i=1}^{n_e} \frac{\phi_i'(0)^2}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}$$

$$= \frac{\frac{1}{J}\prod_{i=1}^{n_e}(s^2 + 2\zeta_i\omega_i s + \omega_i^2) + s^2\sum_{i=1}^{n_e}\phi_i'(0)^2\prod_{j\neq i}^{n_e}(s^2 + 2\zeta_j\omega_j s + \omega_j^2)}{s^2\prod_{i=1}^{n_e}(s^2 + 2\zeta_i\omega_i s + \omega_i^2)}$$

- torque $\tau \mapsto$ tip angle θ_t

$$P_t(s) = \frac{1}{Js^2} + \sum_{i=1}^{n_e} \frac{\phi_i'(0)\,\frac{\phi_i(\ell)}{\ell}}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}$$

$$= \frac{\frac{1}{J}\prod_{i=1}^{n_e}(s^2 + 2\zeta_i\omega_i s + \omega_i^2) + s^2\sum_{i=1}^{n_e}\phi_i'(0)\,\frac{\phi_i(\ell)}{\ell}\prod_{j\neq i}^{n_e}(s^2 + 2\zeta_j\omega_j s + \omega_j^2)}{s^2\prod_{i=1}^{n_e}(s^2 + 2\zeta_i\omega_i s + \omega_i^2)}$$

A numerical example

- physical data

$$J_0 = 0.002, \quad \ell = 1, \quad \rho = 0.5, \quad EI = 1 \quad (m_p = J_p = 0)$$

- by considering up to $n_e = 5$ modes (and no damping), we obtain

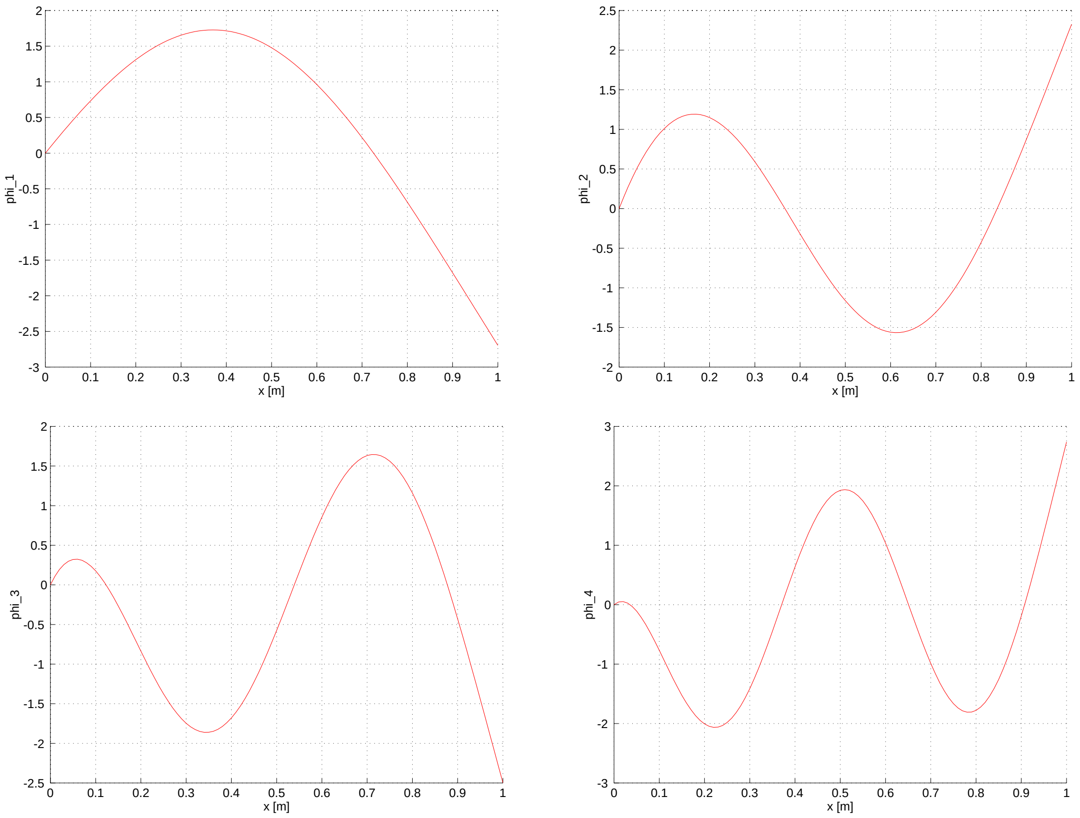
$$\Omega^2 = \text{diag}\,\{421.585,\ 3122.603,\ 10273.194,\ 31562.286,\ 82049.350\}$$

$$\Phi'^T = \left[\ 7.8259 \quad 14.6803 \quad 12.1284 \quad 6.4761 \quad 3.7648\ \right]$$

$$\Phi_\ell^T = \left[\ -2.6954 \quad 2.3268 \quad -2.4970 \quad 2.7380 \quad -2.7982\ \right]$$

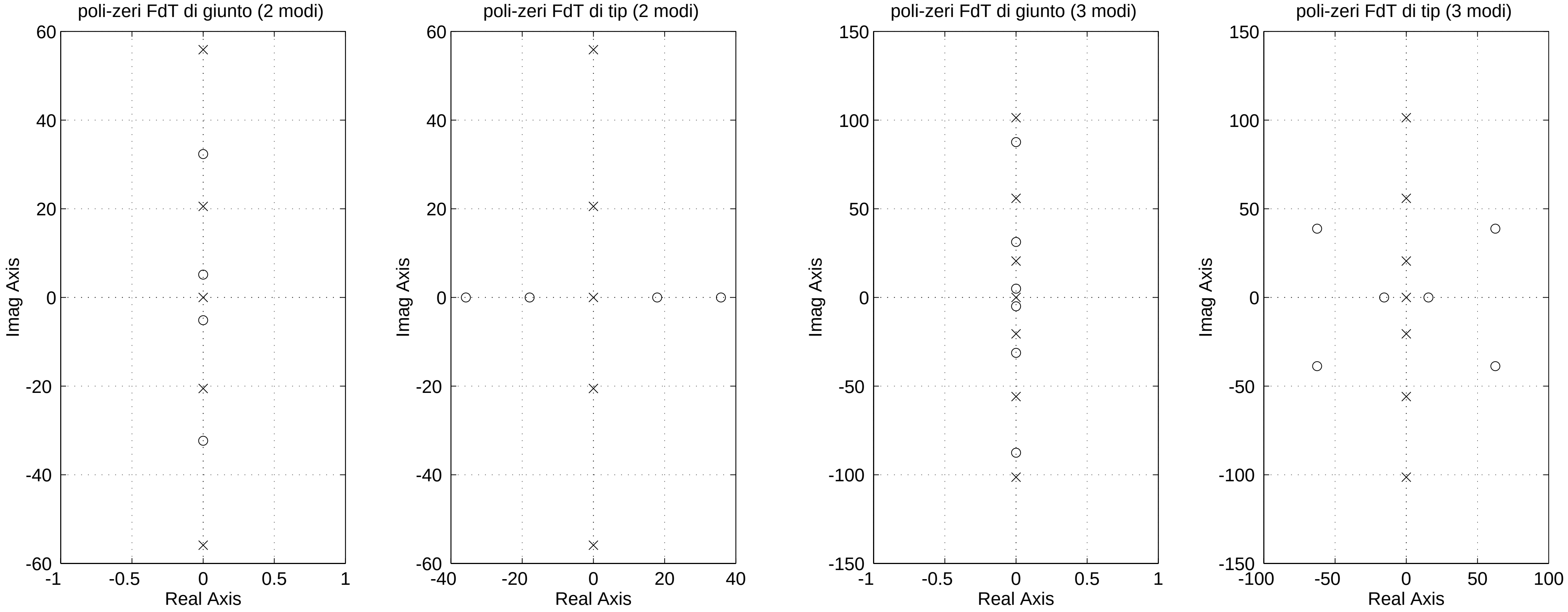
!! note the alternating signs of $\phi_i(\ell)$'s ...

First four mode shapes



at $f_1 = 3.2678$, $f_2 = 8.8936$, $f_3 = 16.1314$, and $f_4 = 28.2751$ [Hz]

Pole-zero patterns

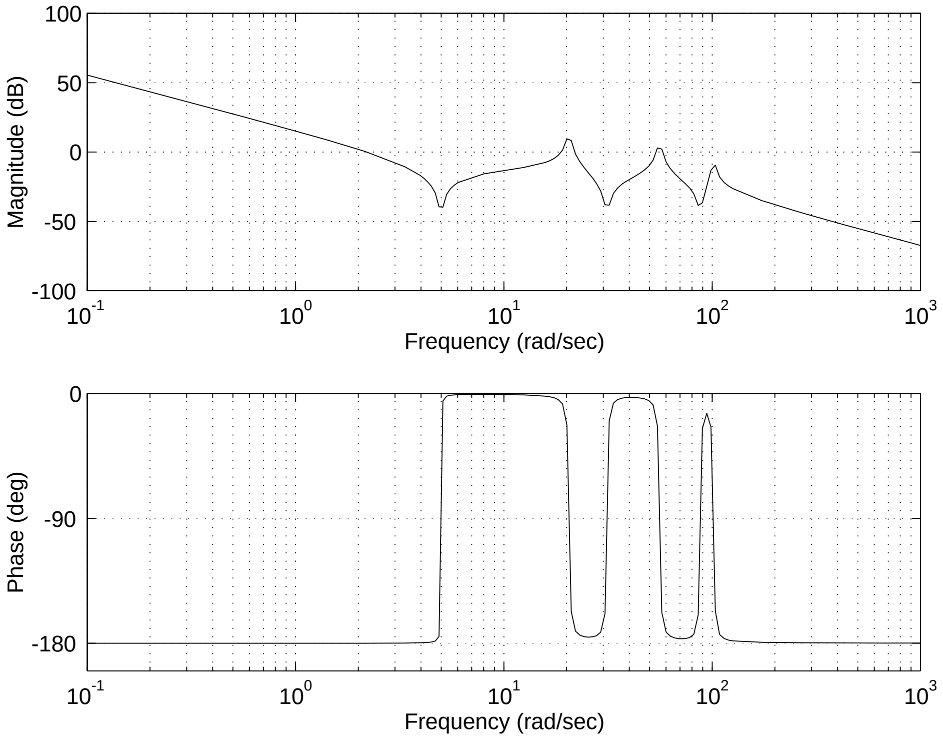


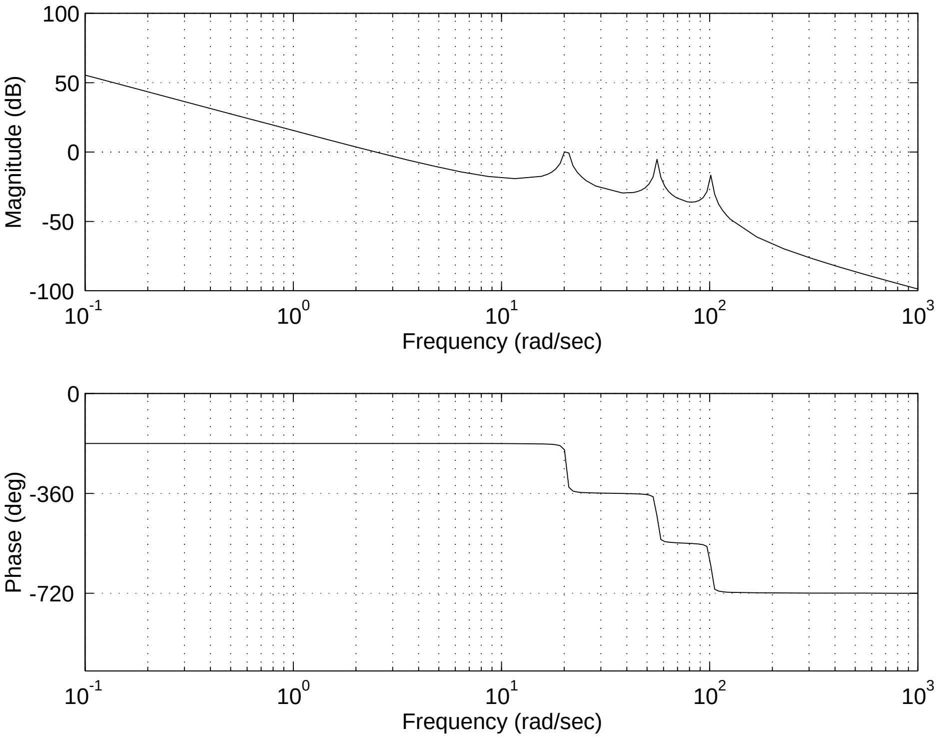
two modes

clamped joint and tip output

three modes

Frequency responses





clamped joint three modes tip

Useful control-oriented remarks

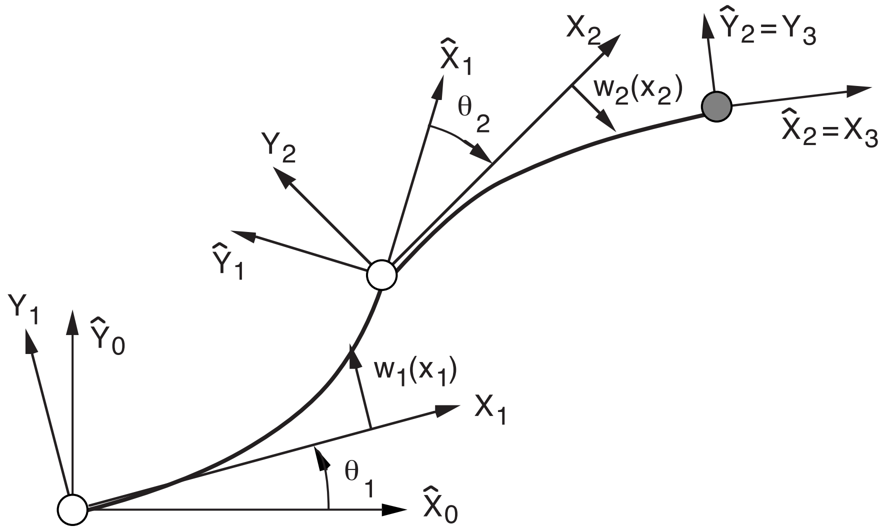
- in pole-zero patterns of $P_c(s)$, zeros precede and alternate with poles on the imaginary axis $\Rightarrow$ passivity property
- zero patterns of $P_t(s)$ are symmetric w.r.t. the imaginary axis $\Rightarrow$ non-minimum phase property $\Rightarrow$ no direct system inversion is feasible
- while moving the output along the link $(P_x(s))$, zeros migrate along imaginary axis and several phenomema occurr:
  - total pole-zero cancellation when pointing at CoM (vibrations unobservable from rigid motion variable θ)
  - for a particular $x^* \in (0, \ell)$, all zeros vanishes together at infinity ($P_{x^*}(s)$ has maximum relative degree equal to $2(n_e + 1)$)
  - beyond x^* (e.g., for $x = \ell$), all pairs of zeros reappear in Re^+/Re^-

- modal damping destroys perfect symmetry in zeros and poles (system is anyway asymptotically stable), but not the non-minimum phase property of $P_t(s)$
- from the Bode plots, it follows that classical controller synthesis in the frequency domain is harder for the tip output
  - multiple crossing of 0dB axis of $|P_t(j\omega)|$ —especially for high control gain
  - increasing phase lag when adding modes

Dynamic modeling of robots with multiple flexible links

- a convenient kinematic description should be used, both for rigid body motion and flexible deformation
- recursive procedures can be set up for open chains with flexible links, similarly to the rigid case
- differential kinematic relationships are needed for computing kinetic and potential energy, within a Lagrangian approach
- modeling results from the one-link case can be embedded (with caution) in the description of each flexible link of the robot
- to limit complexity, we sketch here only the planar case (with gravity) of robots with N flexible links undergoing small bending deformations

Kinematics



- link i: rigid motion by clamped angle $\theta_i(t)$; lateral bending $w_i(x_i, t)$, $x_i \in [0, \ell_i]$
- position vectors and (rigid/flexible) rotation matrices ($w'_{ie} = \left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=\ell_i}$)

$$ {}^i p_i(x_i) = \begin{bmatrix} x_i \\ w_i(x_i) \end{bmatrix} \quad {}^i r_{i+1} = {}^i p_i(\ell_i) \quad A_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \quad E_i = \begin{bmatrix} 1 & -w'_{ie} \\ w'_{ie} & 1 \end{bmatrix} $$

- recursive equations for absolute quantities in $(\hat{X}_0, \hat{Y}_0)$

$$p_i = r_i + W_i {}^i p_i \quad r_{i+1} = r_i + W_i {}^i r_{i+1} \quad W_i = W_{i-1} E_{i-1} A_i$$

- differential kinematics
  - absolute angular velocity of frame (X_i, Y_i)

$$\dot{\alpha}_i = \sum_{j=1}^{i} \dot{\theta}_j + \sum_{k=1}^{i-1} \dot{w}'_{ke}$$

  - absolute linear velocity of a point on link i

$$\dot{p}_i = \dot{r}_i + \dot{W}_i {}^i p_i + W_i {}^i \dot{p}_i$$

where ${}^i \dot{p}_i = [\, 0 \quad \dot{w}_i(x_i)\,]^T$ (link extension is neglected)

Kinetic energy

$$T = \sum_{i=1}^{N} T_{hi} + \sum_{i=1}^{N} T_{\ell i} + T_p$$

- hub i

$$T_{hi} = \frac{1}{2} m_{hi} \dot{r}_i^T \dot{r}_i + \frac{1}{2} J_{hi} \dot{\alpha}_i^2$$

- link i

$$T_{\ell i} = \frac{1}{2} \int_0^{\ell_i} \rho_i(x_i) \dot{p}_i(x_i)^T \dot{p}_i(x_i) dx_i$$

- payload

$$T_p = \frac{1}{2} m_p \dot{r}_{N+1}^T \dot{r}_{N+1} + \frac{1}{2} J_p (\dot{\alpha}_N + \dot{w}'_{Ne})^2$$

Potential energy

$$U = \sum_{i=1}^{N} U_{ei} + \sum_{i=1}^{N} U_{ghi} + \sum_{i=1}^{N} U_{g\ell i} + U_{gp}$$

- elastic energy of link i

$$U_{ei} = \frac{1}{2} \int_0^{\ell_i} (EI)_i(x_i) \left(\frac{d^2 w_i(x_i)}{dx_i^2} \right)^2 dx_i$$

- gravitational energy of hub i and of link i

$$U_{ghi} = -m_{hi} g_0^T r_i \qquad U_{g\ell i} = -g_0^T \int_0^{\ell_i} \rho_i(x_i) p_i(x_i) dx_i$$

- gravitational energy of payload

$$U_{gp} = -m_p g_0^T r_{N+1}$$

where g_0 is the gravity acceleration vector

Euler-Lagrange equations

- introduce any finite-dimensional discretization for $w_i(x_i, t)$

$$w_i(x_i, t) = \sum_{j=1}^{n_{ei}} \varphi_{ij}(x_i)\delta_{ij}(t) \qquad i = 1, \ldots, N$$

- Lagrangian is in terms of $N + M$ generalized coordinates, $M = \sum_{i=1}^{N} n_{ei}$,

$$L = T - U = L\left(\{\theta_i(t), \delta_{ij}(t), \dot{\theta}_i(t), \dot{\delta}_{ij}(t)\}\right)$$

and satifies to

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} &= \tau_i \qquad i = 1, \ldots, N \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}_{ij}}\right) - \frac{\partial L}{\partial \delta_{ij}} &= 0 \qquad j = 1, \ldots, n_{ei} \quad i = 1, \ldots, N \end{aligned}$$

being τ_i the torque delivered by the actuator at joint i

- the general dynamic model of flexible robots (including modal damping) is then

$$\boxed{\begin{bmatrix} M_{\theta\theta}(\theta,\delta) & M_{\theta\delta}(\theta,\delta) \\ M_{\theta\delta}^T(\theta,\delta) & M_{\delta\delta}(\theta,\delta) \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} c_\theta(\theta,\delta,\dot{\theta},\dot{\delta}) \\ c_\delta(\theta,\delta,\dot{\theta},\dot{\delta}) \end{bmatrix} + \begin{bmatrix} g_\theta(\theta,\delta) \\ g_\delta(\theta,\delta) \end{bmatrix} + \begin{bmatrix} 0 \\ D\dot{\delta} + K\delta \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}}$$

with blocks of suitable dimensions (e.g., $M_{\theta\delta}$ in the inertia matrix is $(N \times M)$), or in the more compact form

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) + \begin{bmatrix} 0 \\ D\dot{\delta} + K\delta \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

with $q := (\theta, \delta) \in I\!R^{N+M}$

- vector of centrifugal/Coriolis terms can be factorized using Christoffel symbols

$$c(q,\dot{q}) = C(q,\dot{q})\dot{q} = \begin{bmatrix} C_{\theta\theta}(q,\dot{q}) & C_{\theta\delta}(q,\dot{q}) \\ C_{\delta\theta}(q,\dot{q}) & C_{\delta\delta}(q,\dot{q}) \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\delta} \end{bmatrix}$$

Model properties

- as usual, matrix $\dot{M} - 2C$ is skew-symmetric — also blockwise, e.g. $\dot{M}_{\delta\delta} - 2C_{\delta\delta}$
- spatial dependence in the kinetic and potential energy of the links can be resolved introducing a set of dynamic coefficients so that (De Luca, Siciliano, 1991)

$$Y(\theta, \delta, \dot{\theta}, \dot{\delta}, \ddot{\theta}, \ddot{\delta})a = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

  where constant vector a summarizes the mechanical (rigid + flexible) properties of the links and can be computed off-line (or identified experimentally)

- choice of assumed modes —basis functions $\varphi_{ij}(x_i)$ for bending deformation

– admissible functions satisfy only geometric b.c.'s

– comparison functions (FE method, Ritz-Kantorovich) satisfy also natural b.c.'s

– orthonormal eigenfunctions (links modeled as Euler-Bernoulli beams) leads to simplifications in inertia submatrix $M_{\delta\delta}$ (block diagonal, constant)

- a common approximation evaluates total kinetic energy in the undeformed arm configuration $\delta = 0$

  $\Rightarrow M = M(\theta)$ and thus $c = c(\theta, \dot{\theta}, \dot{\delta})$

  $\Rightarrow c_\delta$ loses quadratic dependence on $\dot{\delta}$

- moreover, if $M_{\delta\delta}$ is constant

  $\Rightarrow c_\theta$ loses quadratic dependence on $\dot{\delta}$

  $\Rightarrow c_\delta$ is a quadratic function of $\dot{\theta}$ *only*

- if also $M_{\theta\delta}$ is constant

  $\Rightarrow c_\delta \equiv 0$

  $\Rightarrow c_\theta$ is a quadratic function of $\dot{\theta}$ *only*

- finally, small deformation of each link implies $g_\delta = g_\delta(\theta)$

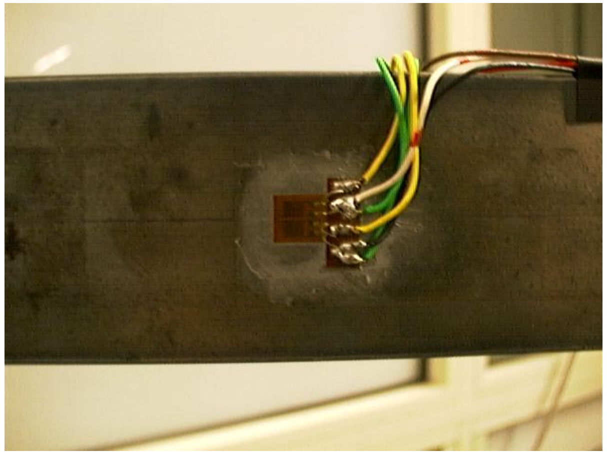
Control problems

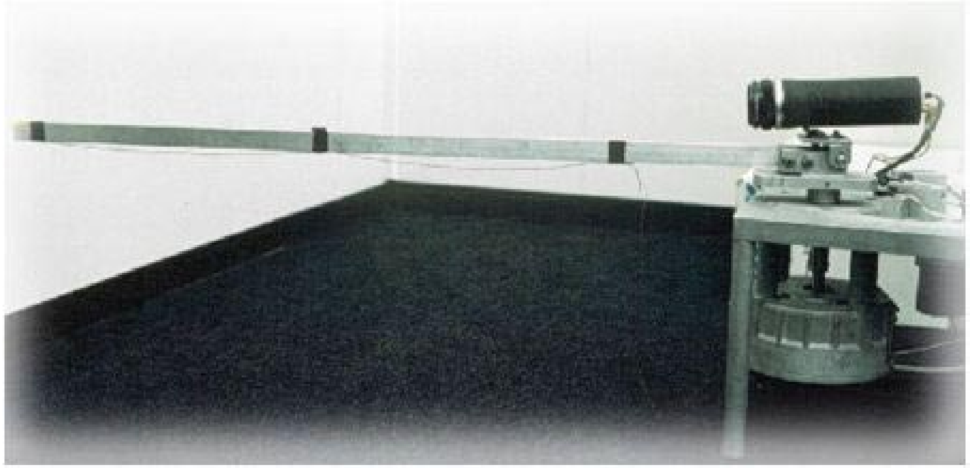
- regulation to a constant equilibrium configuration $(\theta, \delta, \dot{\theta}, \dot{\delta}) = (\theta_d, \delta_d, 0, 0)$
  - only the desired joint position θ_d is assigned, while δ_d has to be determined
  - θ_d may come from the kineto-static inversion of a desired cartesian pose x_d, but no closed-form solution exists (see De Luca, Panzieri, 1994)
  - direct kinematics of FL robots is in fact a complete function of rigid and flexible variables: $x = \text{kin}(\theta, \delta)$
- tracking of a joint trajectory $\theta_d(t)$ —*the easy case*
- tracking of a end-effector trajectory $x_d(t)$ —*the difficult case*
- rest-to-rest motion in given time T (a trajectory planning problem in first place)

∗ in tracking problems, controllers try to stiffen the flexible arm at a point in a way or the other

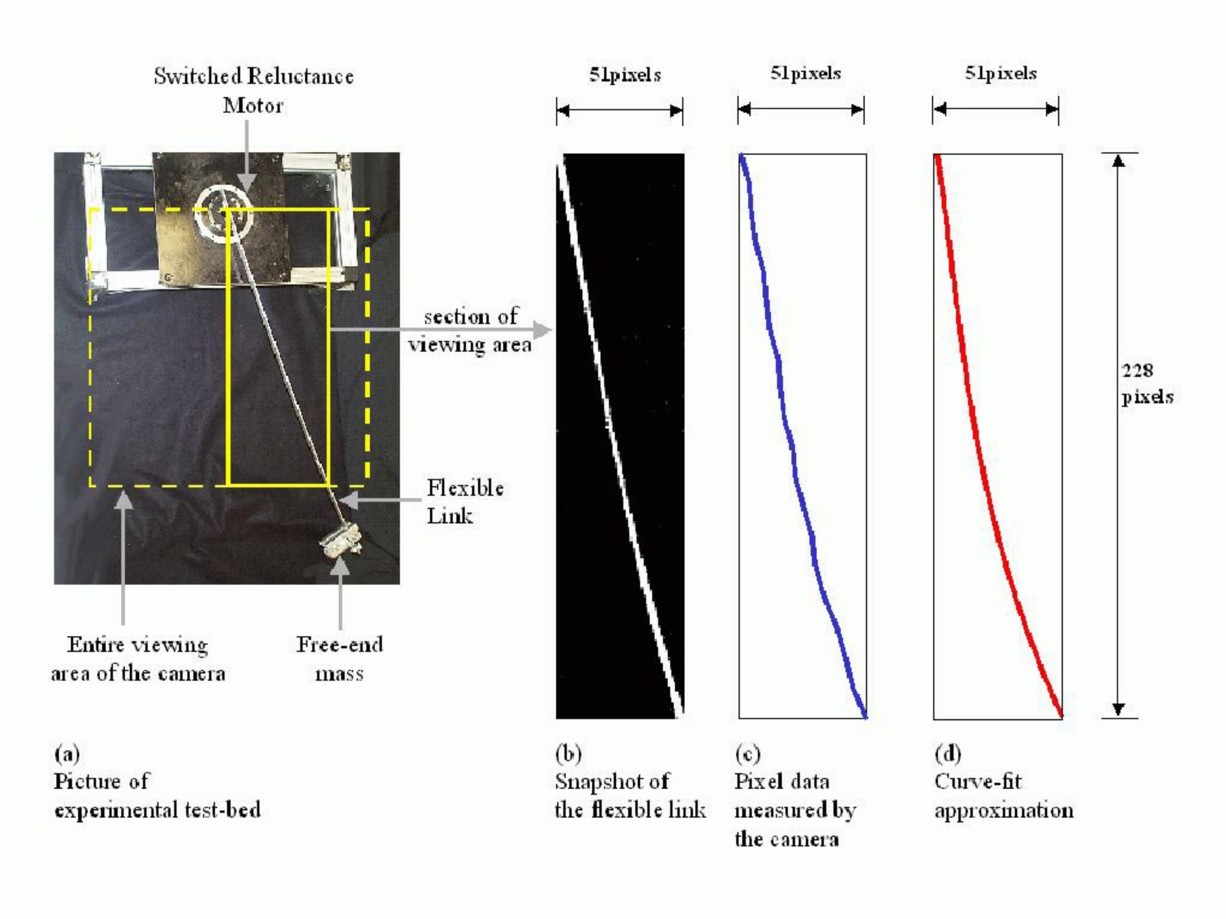
Sensing requirements

- full state feedback requires sensing of joint/motor variables $(\theta, \dot{\theta})$, deflections δ, and deflection rates $\dot{\delta}$ (no direct sensor available)
- at least encoder + tachometer on the motor axis (sometimes is enough . . .)
- a range of sensors for measuring δ (or deformation related quantities), each with pros and cons: strain gauges, accelerometers, optical sensors, video camera (on-board or fixed in workspace), piezoelectric actuating/sensing devices, . . .





- problems with camera: frame rate, field of view/accuracy



Regulation with joint PD + feedforward

- for regulation tasks, consider the control law

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g_\theta(\theta_d, \delta_d)$$

with symmetric (diagonal) $K_P > 0$, $K_D > 0$, and the associated link deflection

$$\delta_d := -K^{-1} g_\delta(\theta_d)$$

$\boxed{\textbf{Theorem}\ \text{(De Luca, Siciliano, 1993)}}$ If

$$\lambda_{min}\left(\begin{bmatrix} K_P & 0 \\ 0 & K \end{bmatrix}\right) > \alpha$$

then the closed-loop equilibrium state $(\theta_d, \delta_d, 0, 0)$ is asymptotically stable

Remarks

- Lyapunov-based proof similar to the joint elastic case (not repeated here)
- determination of α
  - in view of small deformation

$$U_e = \frac{1}{2}\delta^T K \delta \leq U_{e,max} \quad \Rightarrow \quad \|\delta\| \leq \sqrt{\frac{2U_{e,max}}{\lambda_{max}(K)}}$$

  - bound on the gradient of the gravitational term

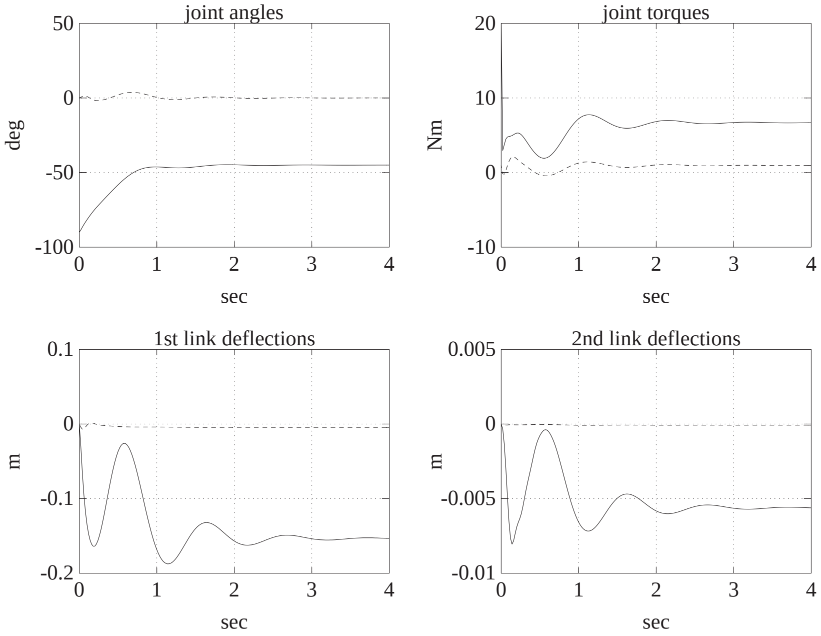
$$\left\|\frac{\partial g}{\partial q}\right\| \leq \alpha_0 + \alpha_1\|\delta\| \leq \alpha_0 + \alpha_1\sqrt{\frac{2U_{e,max}}{\lambda_{max}(K)}} =: \alpha$$

- in absence of modal damping $D = 0$, special care in LaSalle analysis
- for tip regulation, compute θ_d by solving via iterative techniques

$$\text{kin}\left(\theta, -K^{-1}g_\delta(\theta)\right) = x_d$$

Numerical results

- a two-link flexible arm with two bending modes for each link with $f_{11} = 1.4$, $f_{12} = 5.1$, $f_{21} = 5.2$, $f_{22} = 32.4$ [Hz]
- point-to-point motion: $\theta(0) = (-90^\circ, 0) \rightarrow \theta_d = (-45^\circ, 0)$



Joint trajectory tracking

- given a desired $\theta_d(t) \in C^2$, assuming that the state is measurable and the dynamic model of FL robot is available, we proceed by system inversion from the joint position output θ
- a nonlinear static state feedback is obtained that decouples and linearizes the input-output behavior, leaving an unobservable internal (nonlinear) dynamics
- exponential stabilization of the output tracking error is performed on the linear side of the problem
- some stability/boundedness of the internal system dynamics should be enforced
- original results in (De Luca, Siciliano, 1993b)

System inversion

- from second set of M equations in the dynamic model, solve (globally) for $\ddot{\delta}$

$$\ddot{\delta} = -M_{\delta\delta}^{-1}(c_\delta + g_\delta + K\delta + D\dot{\delta} + M_{\theta\delta}^T\ddot{\theta})$$

  and plug in first set of N equations $\Rightarrow$ effects of flexible dynamics onto rigid dynamics

$$\left(M_{\theta\theta} - M_{\theta\delta}M_{\delta\delta}^{-1}M_{\theta\delta}^T\right)\ddot{\theta} + c_\theta + g_\theta - M_{\theta\delta}M_{\delta\delta}^{-1}\left(c_\delta + g_\delta + K\delta + D\dot{\delta}\right) = \tau$$

- Matrix $M_{\theta\theta} - M_{\theta\delta}M_{\delta\delta}^{-1}M_{\theta\delta}^T$ has always full rank, since

$$\begin{bmatrix} M_{\theta\theta} & M_{\theta\delta} \\ M_{\theta\delta}^T & M_{\delta\delta} \end{bmatrix}\begin{bmatrix} I & 0 \\ -M_{\delta\delta}^{-1}M_{\theta\delta}^T & I \end{bmatrix} = \begin{bmatrix} M_{\theta\theta} - M_{\theta\delta}M_{\delta\delta}^{-1}M_{\theta\delta}^T & M_{\theta\delta} \\ 0 & M_{\delta\delta} \end{bmatrix}$$

- system output θ has uniform vector relative degree $\{2, 2, \dots, 2\}$ ($\ddot{\theta}$ depends on τ in a nonsingular way)
- define the nonlinear control law

$$\tau = \left(M_{\theta\theta} - M_{\theta\delta} M_{\delta\delta}^{-1} M_{\theta\delta}^T\right) a + c_\theta + g_\theta - M_{\theta\delta} M_{\delta\delta}^{-1} (c_\delta + g_\delta + K\delta + D\dot{\delta})$$

in which only the inversion inertia block $M_{\delta\delta}^{-1}$ is required

- the closed-loop system is

$$\boxed{\begin{aligned} \ddot{\theta} &= a \\ \ddot{\delta} &= -M_{\delta\delta}^{-1}\left(M_{\theta\delta}^T a + c_\delta + g_\delta + D\dot{\delta} + K\delta\right) \end{aligned}}$$

- for stabilizing the output tracking error $e = \theta_d - \theta$, choose

$$a = \ddot{\theta}_d + K_D(\dot{\theta}_d - \dot{\theta}) + K_P(\theta_d - \theta)$$

with (diagonal) $K_P > 0$, $K_D > 0$

Analysis of internal dynamics

- zero dynamics, when output $\theta(t) \equiv 0$ (or constant):

$$\ddot{\delta} = -M_{\delta\delta}^{-1} \left(c_\delta + g_\delta + D\dot{\delta} + K\delta \right)$$

  asymptotically stable (via Lyapunov argument) ⇒ whole closed-loop system too

- clamped dynamics, when output $\theta(t) \equiv \theta_d(t)$:

$$\ddot{\delta} = -A_2(t)\dot{\delta} - A_1(t)\delta + f_\delta(t)$$

  where (in the case M independent of δ)

$$\begin{array}{rcl} f_\delta(t) & = & -M_{\delta\delta}^{-1}(\theta_d)(M_{\theta\delta}^T(\theta_d)\ddot{\theta}_d + c_\delta(\theta_d, \dot{\theta}_d) + g_\delta(\theta_d)) \\ A_1(t) & = & M_{\delta\delta}^{-1}(\theta_d)K \\ A_2(t) & = & M_{\delta\delta}^{-1}(\theta_d)D \end{array}$$

  all time-varying functions are bounded ⇒ closed-loop stability is ensured

Remarks on joint trajectory tracking

- the input-output linearization result is the nonlinear/MIMO counterpart of the transfer function $\tau \rightarrow \theta_c$ with minimum phase zeros (stable zero dynamics)
- *the more 'rigid' is the tracking of a desired joint trajectory, the less vibration energy is taken out from (or the more is injected into) the rest of flexible arm!!*
- a nominal feedforward is computed by forward integration of flexible dynamics

$$\ddot{\delta} = -M_{\delta\delta}^{-1}(\theta_d, \delta)(c_\delta(\theta_d, \delta, \dot{\theta}_d, \dot{\delta}) + g_\delta(\theta_d) + D\dot{\delta} + K\delta + M_{\theta\delta}^T(\theta_d, \delta)\ddot{\theta}_d)$$

  from $\delta(0) = \delta_0$, $\dot{\delta}(0) = \dot{\delta}_0 \Rightarrow$ nominal (and bounded) evolution $\delta_d(t)$, $\dot{\delta}_d(t)$

- substitution of $(\theta_d(t), \delta_d(t), \dot{\theta}_d(t), \dot{\delta}_d(t))$ in the expression of the control law (w/out feedback) yields $\tau_d(t)$ and the simple local tracking controller

$$\tau = \tau_d(t) + K_P(\theta_d - \theta) + K_D(\dot{\theta}_d - \dot{\theta})$$

End-effector trajectory tracking

- accurate end-effector trajectory tracking is the toughest control problem for flexible robots
- direct extension of inversion strategies to end-effector output ⇒ closed-loop instabilities
  - linear (single-link) case: non-minimum phase tip transfer function
  - nonlinear (multilink) case: unstable zero dynamics for end-effector motion
- main ideas proposed in the literature:
  - resort to suitable feedforward strategy (non-causal solutions)
  - use feedback, but avoid cancellation (causal solutions)

∗ selection of suitable end-effector trajectories that induce smaller arm deflections is of interest in any case

Inversion in frequency domain

- idea: desired motion trajectory as being part of a periodic profile ⇒ use Fourier transforms (Bayo, 1987)
- single-link flexible arm (with generic variables)

$$\begin{bmatrix} m_{\theta\theta} & m_{\delta\theta}^T \\ m_{\delta\theta} & M_{\delta\delta} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \theta \\ \delta \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

- tip position output

$$y(t) = [\,1 \quad c_e^T\,] \begin{bmatrix} \theta \\ \delta \end{bmatrix}$$

- rewrite in terms of (y, δ)

$$\begin{bmatrix} m_{\theta\theta} & m_{\delta\theta}^T - m_{\theta\theta} c_e^T \\ m_{\delta\theta} & M_{\delta\delta} - m_{\delta\theta} c_e^T \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} y \\ \delta \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

- take bilateral Fourier transforms

$$\ddot{Y}(\omega) = \int_{-\infty}^{\infty} \exp(j\omega t)\ddot{y}(t)dt \qquad \ddot{\Delta}(\omega) = \int_{-\infty}^{\infty} \exp(j\omega t)\ddot{\delta}(t)dt$$

$$T(\omega) = \int_{-\infty}^{\infty} \exp(j\omega t)\tau(t)dt$$

and obtain

$$\begin{bmatrix} m_{\theta\theta} & m_{\delta\theta}^T - m_{\theta\theta}c_e^T \\ m_{\delta\theta} & M_{\delta\delta} - m_{\delta\theta}c_e^T + \frac{1}{j\omega}D - \frac{1}{\omega^2}K \end{bmatrix} \begin{bmatrix} \ddot{Y}(\omega) \\ \ddot{\Delta}(\omega) \end{bmatrix} = \begin{bmatrix} T(\omega) \\ 0 \end{bmatrix}$$

- solve for accelerations

$$\begin{bmatrix} \ddot{Y}(\omega) \\ \ddot{\Delta}(\omega) \end{bmatrix} = \begin{bmatrix} g_{11}(\omega) & g_{12}^T(\omega) \\ g_{21}(\omega) & G_{22}(\omega) \end{bmatrix} \begin{bmatrix} T(\omega) \\ 0 \end{bmatrix}$$

- torque is obtained through inversion (in the frequency domain)

$$T(\omega) = \frac{1}{g_{11}(\omega)} \ddot{Y}(\omega) = r(\omega)\ddot{Y}(\omega)$$

- for a given zero-mean $\ddot{y}_d(t)$, with $\ddot{y}_d(t) = 0$ for $t \leq -T/2$ and $t \geq T/2$, this can be embedded into a periodic signal from $(-\infty, +\infty)$

- $\ddot{y}_d(t) \rightarrow \ddot{Y}_d(\omega) \rightarrow T_d(\omega) \rightarrow \tau_d(t)$ from finite inverse Fourier transform

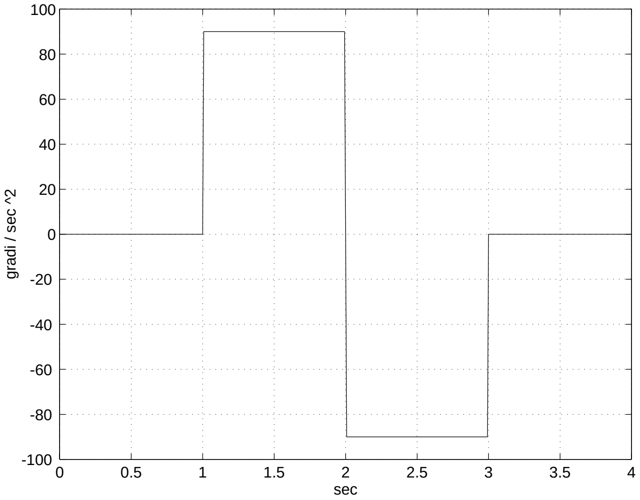
$$\tau_d(t) = \int_{-\infty}^{\infty} r(t-\tau)\ddot{y}_d(\tau)d\tau = \int_{-T/2}^{T/2} r(t-\tau)\ddot{y}_d(\tau)d\tau$$

  expanding beyond $[-T/2, T/2]$ (non-causal inverse system)

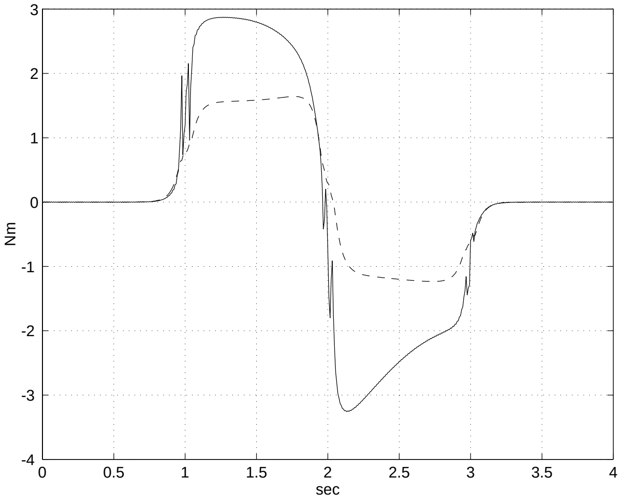
Remarks

- outside the given interval T of output motion, the computed input torque has a
  - precharging action, bringing internal flexible state from rest to a suitable initial state at $t = -T/2$
  - discharging action, bringing internal flexible state from the final state at $t = T/2$ to rest
- *obtained initial condition is the unique state from which inversion control does lead to bounded internal evolution for the desired end-effector output trajectory!*
- truncations (in time or frequency domain) inherent to actual computations (FFT)
- can be extended to the nonlinear (multilink flexible) setting, by repeated linear approximations along nominal trajectory (starting from rigid body motion)

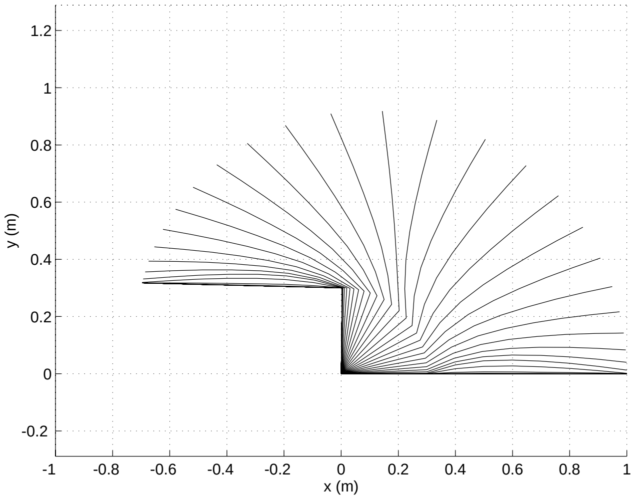
Extension to nonlinear case: End-effector bang-bang trajectory



acceleration profiles

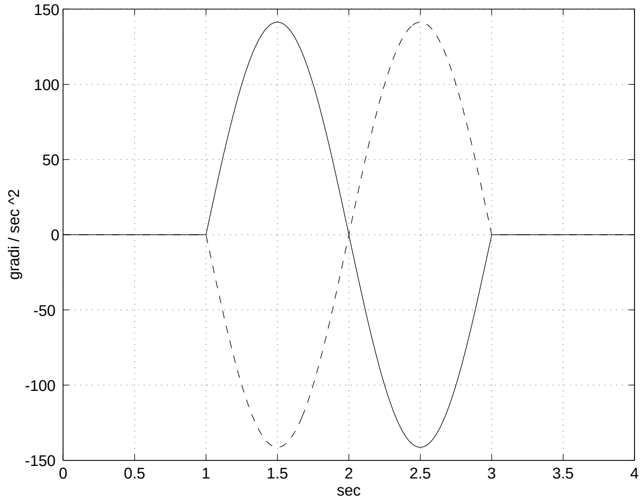


command torques

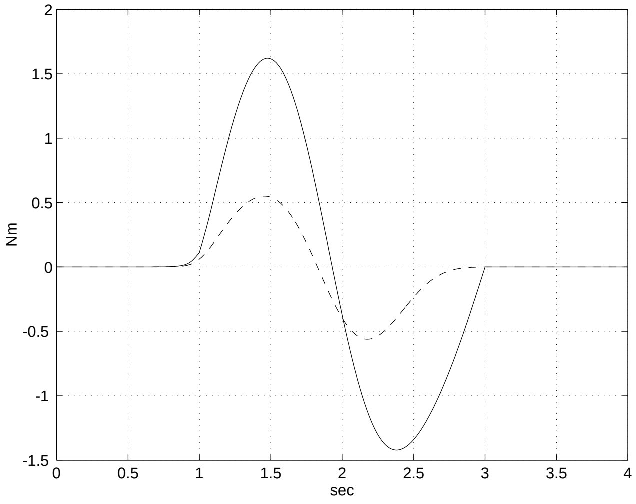


FLEXARM motion

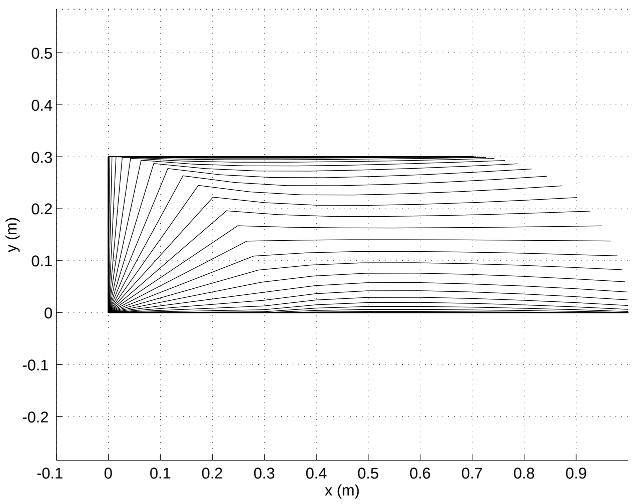
Extension to nonlinear case: End-effector sinusoidal trajectory



acceleration profiles



command torques



FLEXARM motion

Regulation theory

- end-effector trajectory tracking in robots with flexible links is an instance of asymptotic output tracking with internal state stability (regulation problem)
- well-established technique in linear case and, by now, also in nonlinear case
- idea: compute the (bounded!) state trajectory associated to the desired output trajectory (generated by an autonomous antistable system, the exosystem)
- in linear case, write state-space equations (with $x = (q, \dot{q})$) for the flexible arm

$$\dot{x} = Ax + B\tau \qquad e = y - y_d$$

and for the generator of desired output

$$\dot{w} = Sw \qquad y_d = -Qw$$

- when (A, B) is stabilizable, the problem has a solution if and only if the following regulator equations are solvable in Π and Γ

$$\Pi S = A\Pi + B\Gamma \qquad C\Pi + Q = 0$$

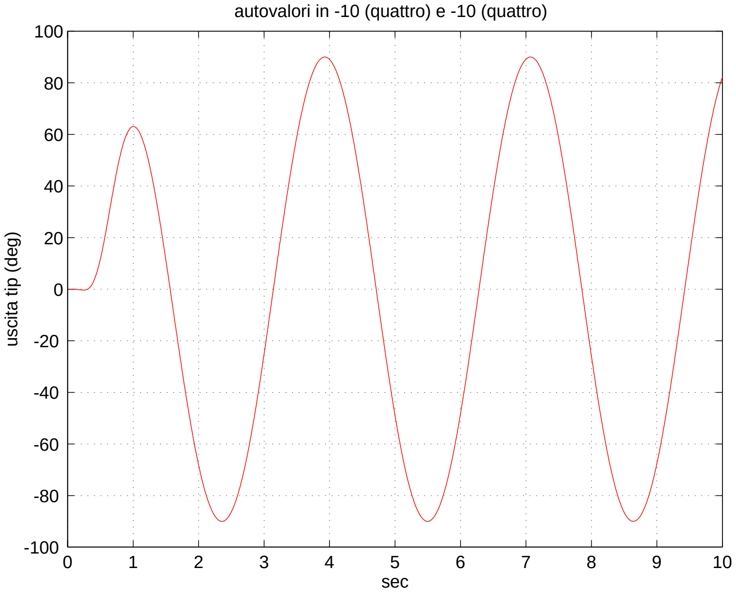
- a state feedback + feedforward controller is then

$$\tau = F(x - \Pi w) + \Gamma w$$

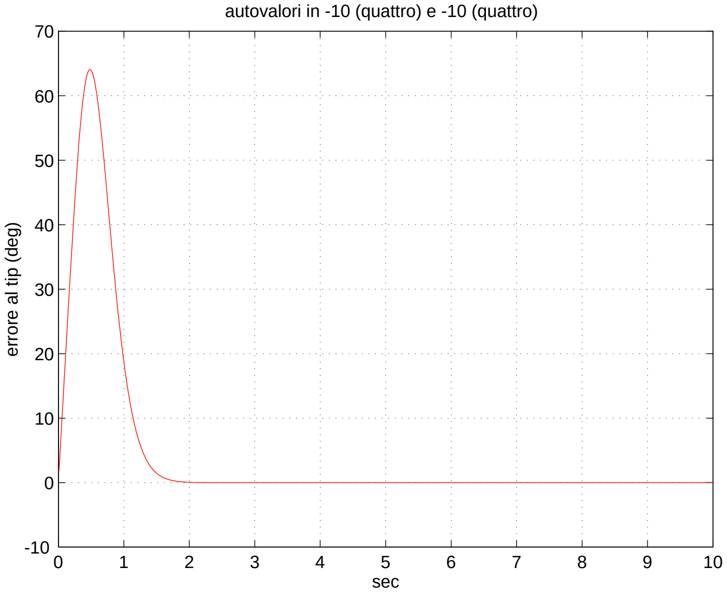
  with feedback matrix F such that $(A + BF)$ is Hurwitz

- the computed Πw is the desired state trajectory; $x_d(0) = \Pi w(0)$ *is the unique initial state from which inversion control does lead to bounded internal evolution!*
- control solutions with dynamic output feedback are also available

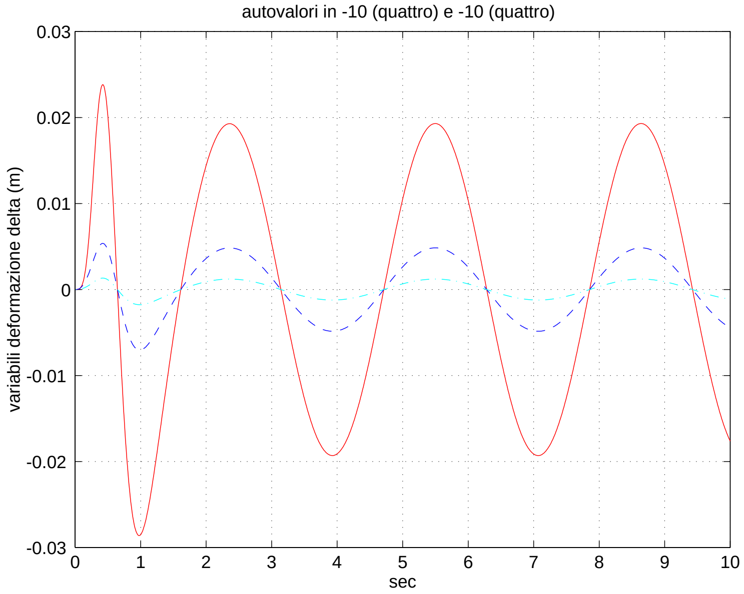
Numerical results for sinusoidal end-effector trajectory



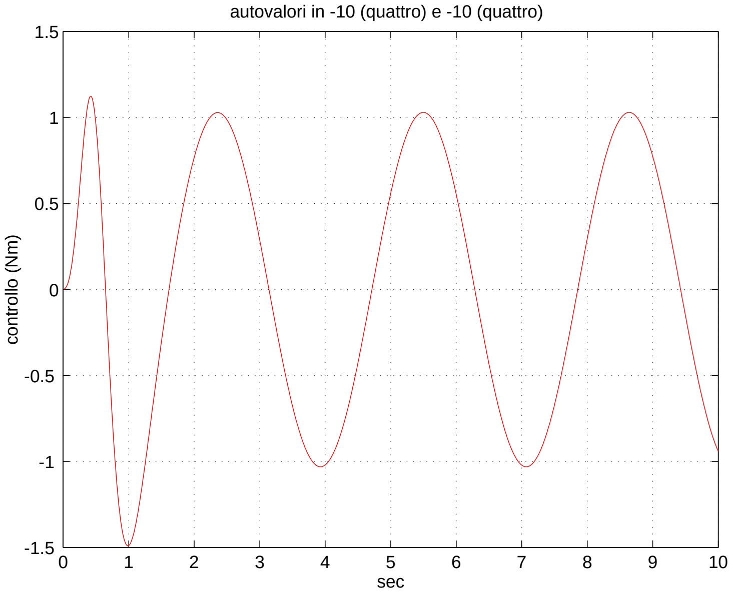
tip output



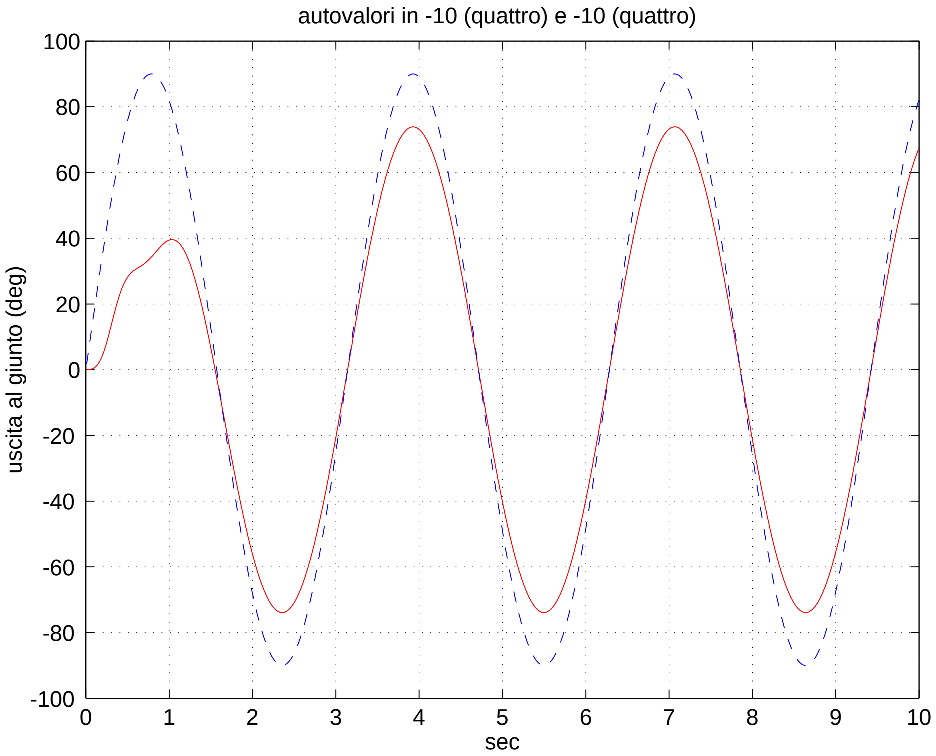
tip error



deformation variables



torque input



clamped joint angle (and desired tip output)

Rest-to-rest motion

- task: execute a slew motion with a FL robot arm between two undeformed configurations in given time
- problem: fast transfers induce residual oscillations extending the actual task completion time
- strategy: design a suitable output and plan output trajectories (and associated torque profiles) inducing complete absence of vibrations at the desired final time
- solution: output with maximum relative degree (no zeros); closed-form algorithm in the linear case; direct extension to MIMO nonlinear case (DFL or flat output, no zero dynamics)

Time-based algorithm for a single flexible link

- choose a parametric output y (unknown c_i's) (De Luca, Di Giovanni, 2001)

$$y = \theta + \sum_{i=1}^{n_e} c_i \delta_i = \theta + c^T \delta$$

- impose τ-independence of (even) output derivatives

$$\ddot{y} = (\frac{1}{J} + \sum_{i=1}^{n_e} c_i \phi_i'(0))\tau - \sum_{i=1}^{n_e} c_i \omega_i^2 \delta_i \qquad \Rightarrow \qquad \boxed{\sum c_i \phi_i'(0) = -1/J}$$

$$y^{[4]} =: \frac{d^4 y}{dt^4} = -\sum_{i=1}^{n_e} c_i \omega_i^2 \phi_i'(0)\,\tau + \sum_{i=1}^{n_e} c_i \omega_i^4 \delta_i \quad \Rightarrow \quad \boxed{\sum c_i \omega_i^2 \phi_i'(0) = 0}$$

and so on, until a set of n_e equations are generated (torque τ appears in the $2(n_e + 1)$-th output derivative)

- solve for the coefficients $c = (c_1, \ldots, c_{n_e})$

$$V \cdot \mathrm{diag}\{\phi_1'(0), \ldots, \phi_{n_e}'(0)\}\, c = [\,-1/J\ 0 \ldots 0\,]^T$$

with Vandermonde matrix V generated by $(\omega_1^2, \ldots, \omega_{n_e}^2)$

- nominal torque $\tau_d(t)$ computed by inversion on highest derivative imposing

$$y^{[2(n_e+1)]} = y_d^{[2(n_e+1)]}$$

for a suitably planned output trajectory $y_d(t)$, $t \in [0, T]$ (given trasfer time)

- for the output trajectory $y_d(t)$, solve a simple interpolation problem

$$y_d(0) = \theta_i \quad y_d(T) = \theta_f \quad \frac{d^i y_d}{dt^i}(0) = \frac{d^i y_d}{dt^i}(T) = 0 \quad i = 1, \ldots, 2n_e + 1$$

e.g., a polynomial of degree $4n_e + 3$ will be sufficient

Laplace-based algorithm (only in the linear case)

- impose that the transfer function has no zeros

$$\frac{y(s)}{\tau(s)} = \frac{1}{Js^2} + \sum_{i=1}^{n_e} \frac{c_i \phi_i'(0)}{s^2 + \omega_i^2} \triangleq \frac{K}{s^2 \prod_{i=1}^{n_e} (s^2 + \omega_i^2)}$$

- partial fractions expansion yields closed-form expressions

$$K = \frac{1}{J} \prod_{i=1}^{n_e} \omega_i^2 \qquad c_i = -\frac{1}{J\phi_i'(0)} \prod_{\substack{j=1 \\ j \neq i}}^{n_e} \frac{\omega_j^2}{\omega_j^2 - \omega_i^2} \qquad (i = 1, \ldots, n_e)$$

- set $y = y_d$ and invert in the transformed domain (then back to time $\longrightarrow \tau_d(t)$)

$$\tau_d(s) = \frac{J}{\prod_{i=1}^{n_e} \omega_i^2} \left[s^2 \prod_{i=1}^{n_e} (s^2 + \omega_i^2) \right] y_d(s)$$

Remarks

- method applies to any linear (controllable) model of a single-link flexible arm
- output structure for modal damping (De Luca, Caiano, Del Vescovo, 2003)

$$y = \theta + \sum_{i=1}^{n_e} c_i \delta_i + \gamma \dot{\theta} + \sum_{i=1}^{n_e} d_i \dot{\delta}_i$$

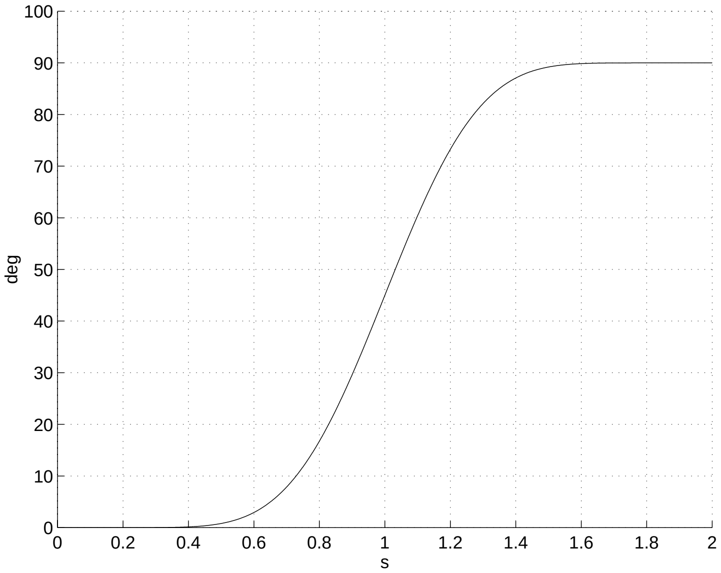
- design output is (in the limit) a specific point x^* on the physical beam: for a given n_e, $c_i = \phi_i(x^*_{n_e})/x^*_{n_e}$ while $\lim_{n_e \to \infty} x^*_{n_e} = x^*$
- for improved torque/time performance, modified method generates smoothed bang-bang/bang-coast-bang torque profiles, with polynomial interpolating phases
- trajectory planning (feedforward) combined with feedback control

$$\tau = \tau_d(t) + K_P(\theta_{c,d}(t) - \theta_c) + K_D(\dot{\theta}_{c,d}(t) - \dot{\theta}_c)$$

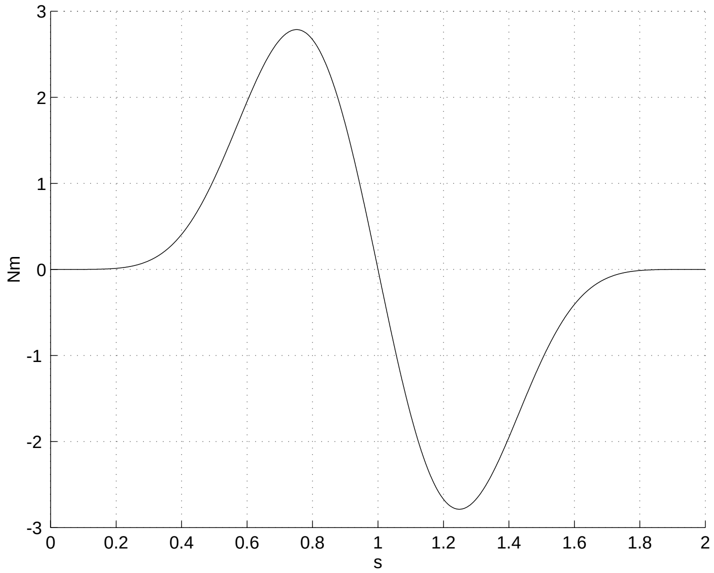
with clamped joint reference $\theta_{c,d}(t)$ computed from the algorithm

Numerical results

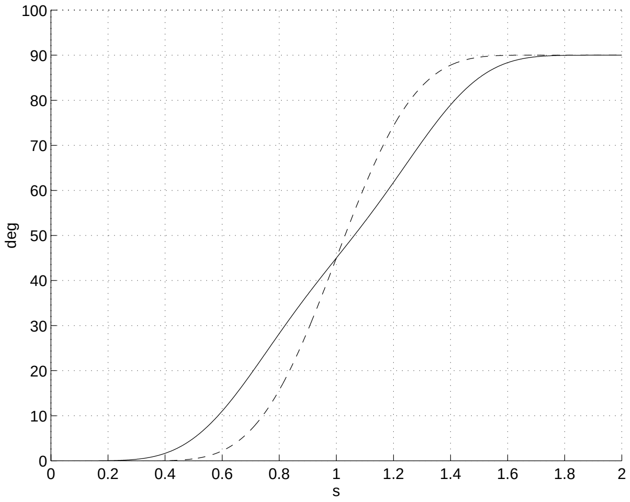
- $n_e = 3$ modes with $f_1 = 4.05$, $f_2 = 12.34$, $f_3 = 22.87$ [Hz]
- $\theta_f - \theta_i = 90°$ in $T = 2$ s
- 19*th* degree polynomial (continuous torque derivative)



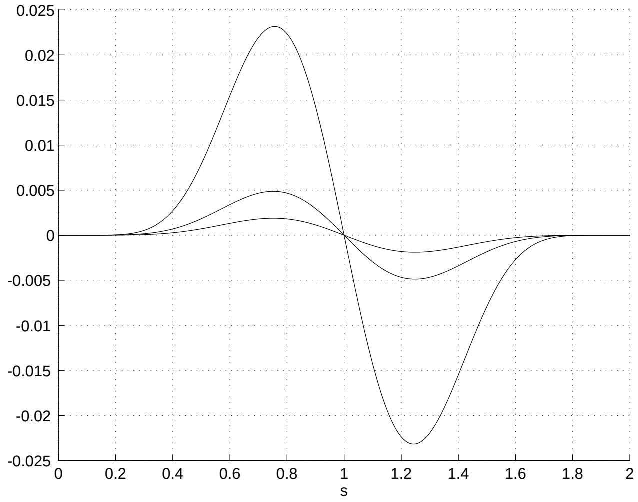
output trajectory



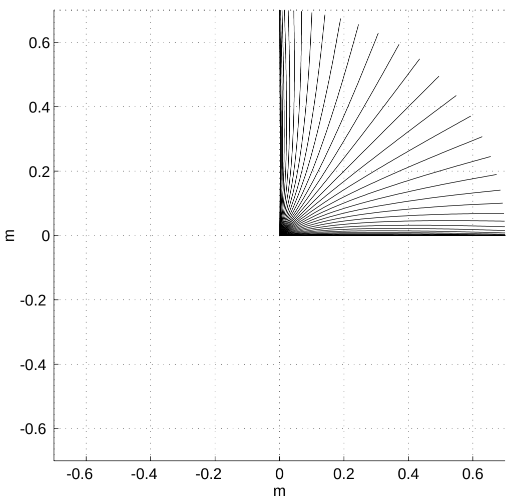
rest-to-rest torque



clamped (—), tip angle (- -)



deformation variables

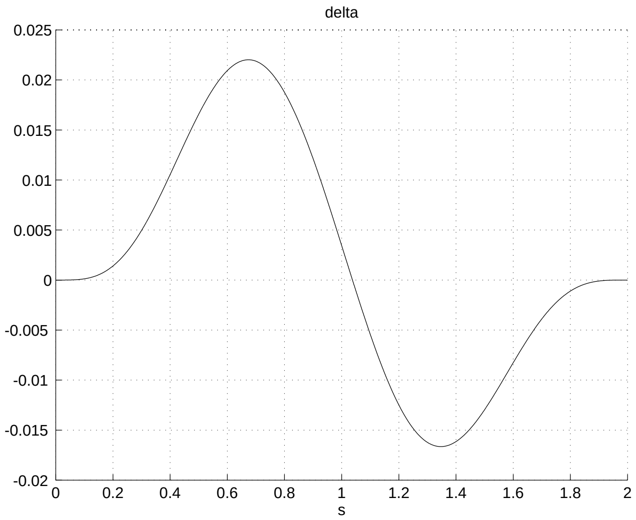


arm motion

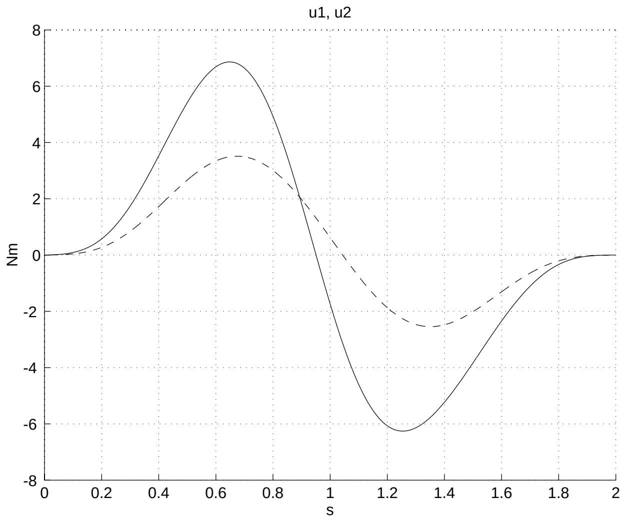
Experiment on DMA single flexible link

Extension to nonlinear case: Rest-to-rest motion

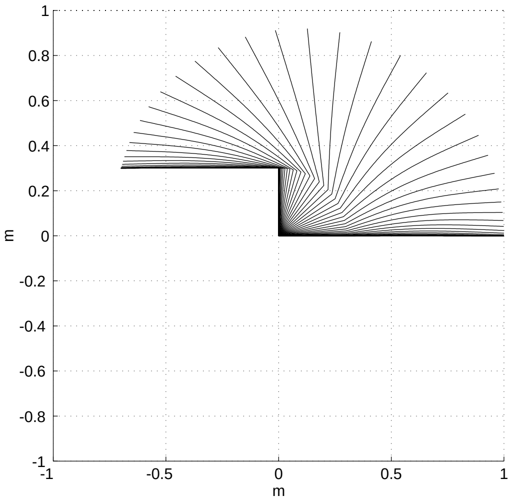
- time-based algorithm for two-link with flexible forearm (De Luca, Di Giovanni, 2001b)



first deformation mode



command torques



FLEXARM motion

Conclusions

- extra effort in dynamic modeling pays off
  - model-based controllers for accurate trajectory tracking
  - proof of stability for model-inpedendent regulation controllers
- conventional control strategies tend to suppress vibrations wherever they arise
  - outcome of the analysis: controlled system should be brought to a vibratory behavior compatible with the given output task
- EJ robots are similar to FL robots in mechanical modeling, but intrinsically different from the control point of view

What has been left out . . .

- singular perturbation modeling and control (for joint or link stiffness $K \longrightarrow \infty$), including corrective and invariant manifold controllers
- iterative learning control that yields same accuracy (for all types of tasks) without using a dynamic model but assuming repetitive tasks
- model uncertainties, disturbances, . . .

References

(Bayo, 1987) "A finite-element approach to control the end-point motion of a single-link flexible robot," JRS, 4, 63–75

(De Luca, Caiano, Del Vescovo, 2003) "Experiments on rest-to-rest motion of a flexible arm," in Experimental Robotics VIII (Siciliano, Dario Eds), STAR 5, Springer, 338–349

(De Luca, Di Giovanni, 2001) "Rest-to-rest motion of a one-link flexible forearm," IEEE/ASME AIM 01, 923–928

(De Luca, Di Giovanni, 2001b) "Rest-to-rest motion of a two-link robot with a flexible forearm," IEEE/ASME AIM 01, 929–935

(De Luca, Lanari, Ulivi, 1991) "End-effector trajectory tracking in flexible arms: Comparison of approaches based on regulation theory," in Advanced Robot Control (Canudas de Wit Ed), LNCIS 162, Springer, 190–206

(De Luca, Panzieri, 1994) "An iterative scheme for learning gravity compensation in flexible robot arms," Automatica, 30(6), 993–1002

(De Luca, Panzieri, Ulivi, 1998) “Stable inversion control for flexible link manipulators,” IEEE ICRA, 799–805

(De Luca, Siciliano, 1991) “Closed-form dynamic model of planar multi-link lightweight robots,” IEEE SMC, 21(4), 826–839

(De Luca, Siciliano, 1993) “Regulation of flexible arms under gravity,” IEEE TRA, 9(4), 463–467

(De Luca, Siciliano, 1993b) “Inversion-based nonlinear control of robot arms with flexible links,” AIAA JGCD, 16(6), 1169–1176

(De Luca, Siciliano, 1996) “Flexible links,” in Theory of Robot Control (Canudas de Wit, Siciliano, Bastin Eds), Springer, 219–261